Ratios, Percentages, and Proportions

Science Learning Center

University of Michigan-Dearborn

Objectives

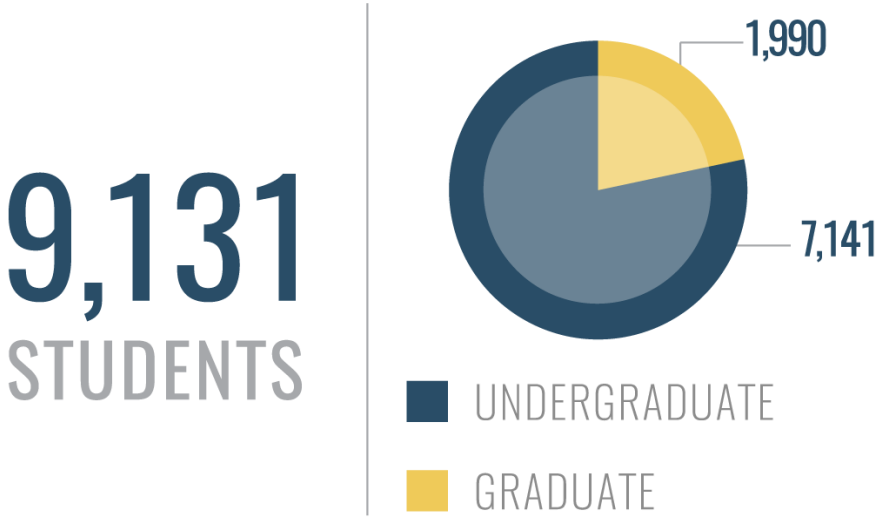
After working through this module, we expect that you will be able to:

- explain the meaning of ratio, percentage, and proportion
- add, subtract, multiply, and divide fractions
- convert between fractions, decimals and percentages
- solve problems involving ratios, percentages, and proportions

Introduction

Ratios, percentages, and proportions are all concerned with dividing something into parts.

For example, we could divide the student body at UM-Dearborn into undergraduate and graduate students.



We can then compare the parts to the whole or compare the parts to each other.

Ratios

A ratio is just a comparison of two values.
(Say, undergraduate and graduate students at UM-Dearborn)

Ratios can be represented in different ways. The ratio of **undergraduate** students to **all** students at UM-Dearborn is:

- In words 7,141 to 9,131
- With a ":" 7,141 : 9,131
- As a fraction 7,141 / 9,131

Ratios can compare parts to the whole, as above, or compare parts to each other. For example, the ratio of undergraduates to graduate students is:

- 7,141 to 1,990 OR 7,141:1,990 OR 7,141/1,990

Fraction Review

- Mathematically, we just treat ratios (and percentages, and proportions) as fractions. Let's do a quick review of fractions!
- Any fraction can be seen as repetition of a "unit":

$\frac{4}{5}$ is shorthand for $\frac{1}{5}+\frac{1}{5}+\frac{1}{5}+\frac{1}{5}$ or $4*\frac{1}{5}$

$\frac{4}{5}$ The numerator counts the number of units

The denominator describes the size of the units (as splits of one)

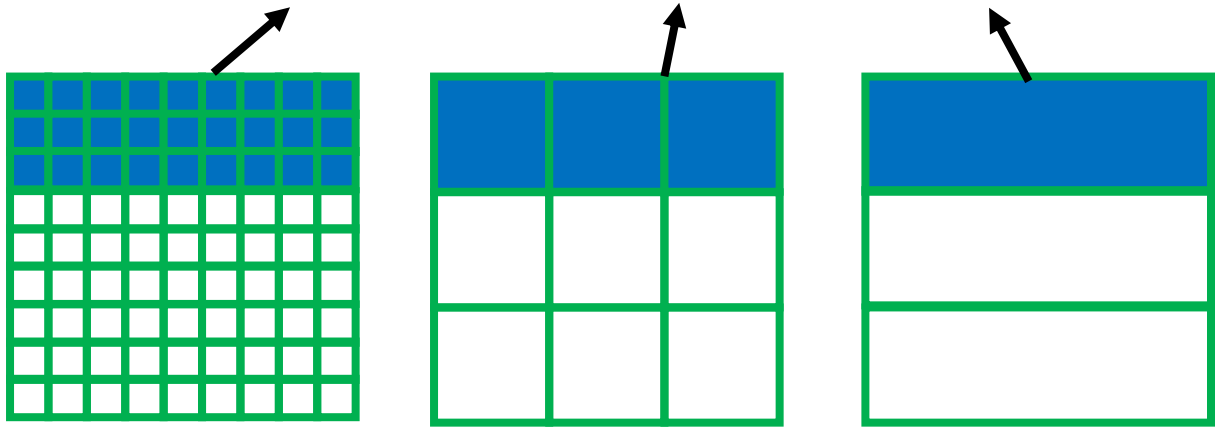
- All fractions have an infinite number of equivalent fractions:

$$\frac{4}{5}*\frac{2}{2}=\frac{8}{10}$$

Fraction Review: Simplifying

- We generally want to simplify fractions as far as possible.
- We can reduce a fraction if both the numerator and denominator can be divided by the same number evenly:

$$\frac{8}{10}\ \begin{matrix}\div 2\\ \div 2\end{matrix} = \frac{4}{5} \qquad \frac{27}{81}\ \begin{matrix}\div 9\\ \div 9\end{matrix} \rightarrow \frac{3}{9}\ \begin{matrix}\div 3\\ \div 3\end{matrix} \rightarrow \frac{1}{3}$$



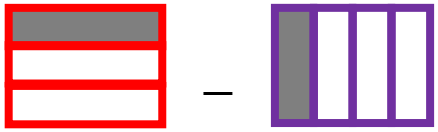
Fraction Review: Addition and Subtraction

- In order to **add or subtract**, fractions must have **common denominators** (same bottom number).

- If they don't start out with a common denominator as in step 1, we must change them to equivalent fractions with a common denominator.

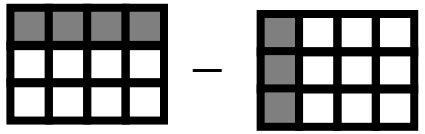
- The easiest way to do this is to multiply the top and bottom of each fraction by the denominator of the other as shown in step 2.

Step 1: $\frac{1}{3} - \frac{1}{4}$

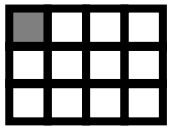


Step 2: $\frac{1}{3} \frac{*4}{*4} - \frac{1}{4} \frac{*3}{*3}$

Step 3: $\frac{4}{12} - \frac{3}{12}$



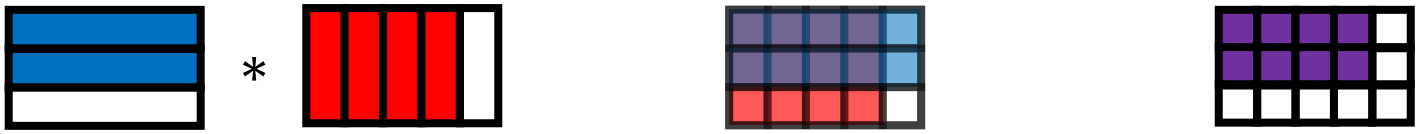
Result: $\frac{1}{12}$



Fraction Review: Multiplication

- When **multiplying** fractions, we don't need a common denominator.
- We simply multiply the numerators and multiply the denominators separately.
- Note that the word "**OF**" means **multiplication**:

$$\frac{2}{3}\ \mathbf{OF}\ \frac{4}{5} \rightarrow \frac{2}{3} * \frac{4}{5} \rightarrow \frac{2 * 4}{3 * 5} \rightarrow \frac{8}{15}$$

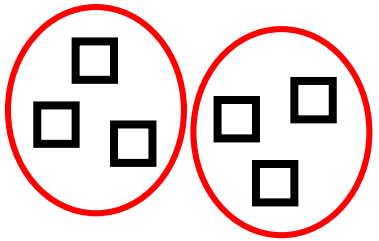


Fraction Review: Division

Dividing fractions often trips people up, so we'll go in-depth…

- Division asks *"How many groups can we get?"*

$$6 \div 3 = 2$$

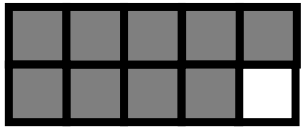


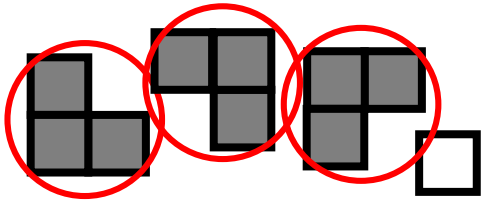
"How many groups of 3 can we get out of 6?" *"We can get 2."*

$$\frac{9}{10} \div \frac{3}{10}$$

"How many groups of $\frac{3}{10}$ *can we get out of* $\frac{9}{10}$*?"*

"We can get 3."





- When the denominators are the **same**, as they are here, you can just divide the numerators: $9 \div 3 = 3$

Fraction Review: Division

- If the denominators are **different**, you can find a common denominator:

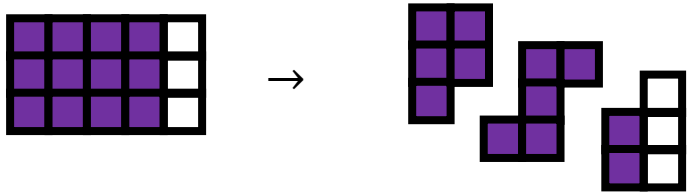
$$\frac{4}{5} \div \frac{1}{3} \rightarrow \frac{4}{5} * \frac{3}{3} \div \frac{1}{3} * \frac{5}{5} \rightarrow \frac{12}{15} \div \frac{5}{15}$$

"How many groups of $\frac{1}{3}$ can we get out of $\frac{4}{5}$?"

- Then just divide the numerators:

$$\frac{12}{15} \div \frac{5}{15} \rightarrow \frac{12}{5}$$ which we can write as $\frac{10}{5} + \frac{2}{5}$ or 2 and $\frac{2}{5}$

"How many groups of $\frac{5}{15}$ can we get out of $\frac{12}{15}$?"



"We can get $\frac{12}{5}$, that is, 2 and $\frac{2}{5}$ groups."

Fraction Review: Division

- A shortcut for this process is to "invert and multiply" or "keep-change-flip". Keep the first fraction, change $\div$ to $*$, and take the reciprocal of the second fraction.

$$\frac{4}{5} \div \frac{1}{3} \rightarrow \frac{4}{5} * \frac{3}{1} \rightarrow \frac{4 * 3}{5 * 1} \rightarrow \frac{12}{5}$$

This shortcut is just a way to skip to the step highlighted below, the equivalent of getting a common denominator and dividing the numerators:

$$\frac{4}{5} \div \frac{1}{3} \rightarrow \frac{4 * 3}{5 * 3} \div \frac{1 * 5}{3 * 5} \rightarrow \frac{12}{15} \div \frac{5}{15} \rightarrow \frac{12}{5}$$

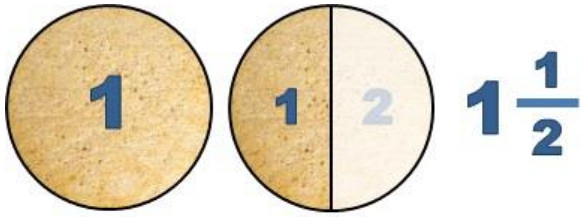
Mixed Fractions

A **mixed fraction** is a whole number and a **fraction** combined into one "**mixed**" number.

Whole Number → $2\frac{2}{5}$ ← Numerator

Denominator

- $1\frac{1}{2} = 1 + \frac{1}{2}$



$1\frac{1}{2}$

- $2\frac{4}{6} = 2 + \frac{4}{6}$

$2\frac{4}{6}$

Mixed Fractions

A **mixed fraction** can be simplified as an *improper fraction,* which is a fraction where the numerator is greater than the denominator.

Example: Write $2\frac{2}{5}$ as an improper fraction.

- Step 1: Separate the whole number and fraction by addition.

$$2\frac{2}{5} = 2 \text{ and } \frac{2}{5} = 2 + \frac{2}{5}$$

- Step 2: Multiply to get the same denominator for both fractions.

$$\frac{2*5}{1*5} + \frac{2}{5} = \frac{10}{5} + \frac{2}{5}$$

- Step 3: Add fractions.

$$\frac{10}{5} + \frac{2}{5} = \frac{10+2}{5} = \boxed{\frac{\mathbf{12}}{\mathbf{5}}}$$

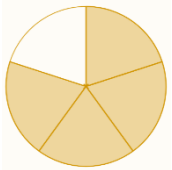
Mixed Fractions

- Therefore,

$$2\frac{2}{5} = \mathbf{\frac{12}{5}}$$

Note: This can be confusing because mixed fractions look similar to multiplication. Generally, if the fraction is being multiplied, it will have a symbol or parenthesis indicating multiplication:

$2 \times \frac{2}{5}$ ***or*** $2 * \frac{2}{5}$ ***or*** $2\left(\frac{2}{5}\right)$ will simplify to $\frac{4}{5}$



$$\frac{12}{5} \neq \frac{4}{5}$$

Practice Problem Set #1

1) $\frac{1}{2} + \frac{3}{4} - \frac{4}{8} = ?$

2) $\frac{4}{\frac{2}{3}} = ?$

3) Teju baked 3 cookies and used $\frac{1}{6}$ cup of cocoa and $\frac{1}{3}$ cup of sugar in total. Assuming she stirred well:

   a. How much cocoa was in each cookie?

   b. How much sugar was in each cookie?

4) Sheetal lives with roommates, and owns $\frac{1}{4}$ of the total groceries in her house. Dina came over as a guest and used $\frac{1}{3}$ of Sheetal's groceries. What fraction of the total groceries in the house did Dina use?

Practice Problem Set #1 Solutions

1) $\frac{3}{4}$

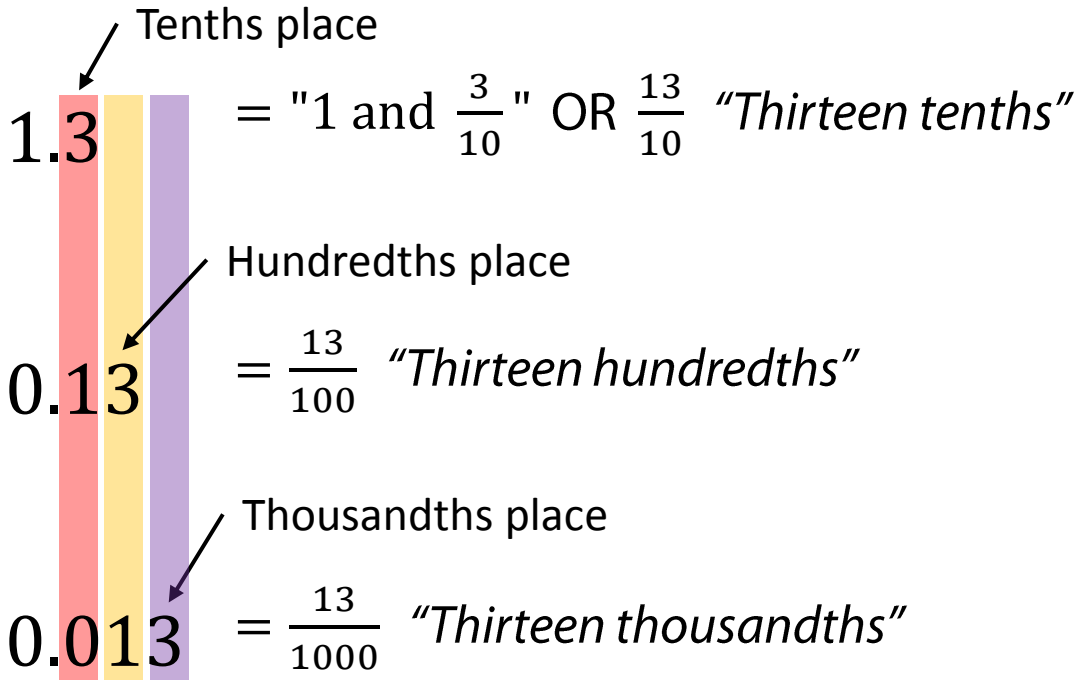
2) 6

3) (a) Cocoa: $\frac{1}{18}$ cup (b) Sugar: $\frac{1}{9}$ cup

3) Dina used $\frac{1}{12}$ of the total groceries in the house.

Decimals as fractions

- To work with ratios, percentages, and proportions, we also need to know how to convert between fractions and decimals.
- A decimal is just a fraction indicated by the place value:



… and so on!

Fractions as decimals

- To convert a fraction to a decimal, you can use:
  - A calculator — "$\frac{3}{4}$ *is 3 divided by 4*"

OR

  - Long division — $\frac{3}{4} \rightarrow 4\overline{)3.00}\ \begin{array}{r}0.75\\ \underline{2\,8}\\ 20\end{array} \rightarrow 0.75$

OR

1. Find a number you can multiply the denominator by to get 10, 100, or any power of 10
2. Multiply both the numerator and denominator by that number
3. Write as a decimal, matching the place value to your new denominator

$$\frac{3}{4} * \frac{25}{25} = \frac{75}{100} = 0.75$$

tenths hundredths

"$\frac{3}{4}$ *is 75 hundredths*"

Percentages as fractions and decimals

- What is a percentage? Just a fraction, with 100 as the denominator! So we can easily convert a percentage to a fraction or a decimal…
- Percent is defined as "per 100" ($\%$ is $\frac{1}{100}$)

  ("Cent" comes from the Latin word for hundred, *centum*)
- So "3 percent" is "3 per 100" OR $\frac{3}{100}$ OR *"3 hundredths"*

$$3\% \rightarrow \frac{3}{100} \rightarrow 0.03$$

- A slightly more interesting example:

$$43.2\% \rightarrow \frac{43.2}{100} \rightarrow \frac{43.2}{100} * \frac{10}{10} \rightarrow \frac{432}{1000} \rightarrow 0.432$$

Decimals as percentages

- It's also straightforward to convert decimals to percentages…
- Percent is defined as "per 100" ($\%$ is $\frac{1}{100}$)
- To go from decimals to percentages, we just multiply by 100%.

  Note that this is the same as multiplying by 1:

$$1 = \frac{100}{100} = 100 * \frac{1}{100} = 100\,\%$$

- For example:

$$0.03 = \frac{3}{100}$$

$$\frac{3}{\cancel{100}} * 1\cancel{00}\,\% = 3\%$$

This makes sense, "3 per 100" is 3%

- A slightly more interesting example:

$$0.432 \rightarrow \frac{432}{1\cancel{000}} * 1\cancel{00}\,\% \rightarrow \frac{432}{10}\,\% \rightarrow 43.2\%$$

Fractions as percentages

- So to convert any fraction to a percentage, we just convert the fraction to a decimal, then convert the decimal to a percentage.

$$\frac{3}{4} \rightarrow \begin{array}{r} 0.75 \\ 4\overline{)3.00} \\ \underline{2\,8} \\ 20 \end{array} \rightarrow 0.75 \rightarrow \frac{75}{100} * 100\% \rightarrow 75\%$$

Calculating with percentages

What are some practical applications of all this?

1. You get a new job making $60,000 per year. They will take out 15% for taxes. How much will you be making per year after taxes?

   - What is 15% **of** $60,000? $15\% * \$60{,}000 \rightarrow \frac{15}{100} * \$60{,}000$

     $15 * \$600 = \$9{,}000$, so they will take out $9,000 for taxes.

   - So you will make $\$60{,}000 - \$9{,}000 = \$51{,}000$ after taxes

2. You bought a shirt because you had a coupon for 20% off. The regular price wasn't marked. You paid $16 for the shirt. What was its regular price?

   - "20% off" means you paid 80% **of** the regular price.
   - $80\% \overset{\text{"of"}}{*} \text{reg. price} = \$16$ so $\text{reg. price} = \frac{\$16}{80\%} \rightarrow$

     $$\text{reg. price} = \frac{\$16}{\frac{80}{100}} \rightarrow \text{reg. price} = \frac{\$16}{1} * \frac{100}{80} \rightarrow$$

     $$\text{reg. price} = \frac{\$160}{8} \rightarrow \text{reg. price} = \$20$$

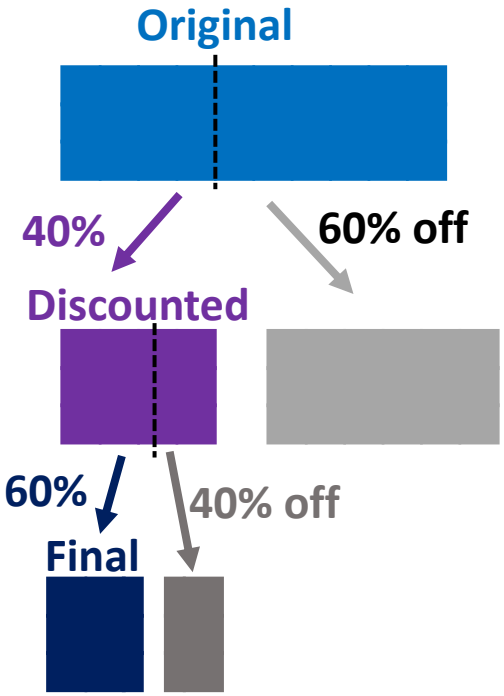
Solving multi-step percentage problems

We can extend these ideas to problems with multiple steps:

After Halloween, a large bag of chocolate ghosts was marked down to 60% off of its original price of $20. You then used a coupon to get 40% off the discounted price. How much did you pay?

Start by clearly writing down what you know:

- "60% off" means the discounted price is 40% **of** the original price
- The final price is 60% **of** the discounted price ("40% off" the discounted price)
- So the final price is 60% **of** 40% **of** the original price



Solving multi-step percentage problems

After Halloween, a large bag of chocolate ghosts was marked down to 60% off of its original price of $20. You then used a coupon to get 40% off the discounted price. How much did you pay?

- The final price is 60% **of** 40% **of** the original price

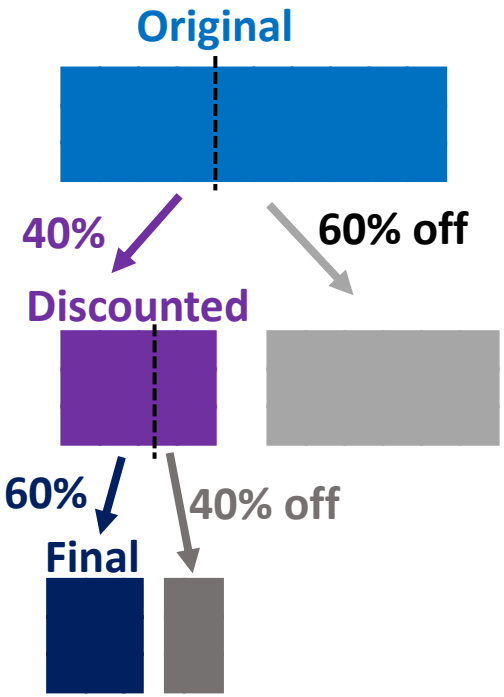
- Remember that "of" means multiplication. So to find a percent of a percent of a value, we take the **original value** and **multiply by each fraction**:

$$\text{Final price} = 60\% \overset{\text{"of"}}{*} 40\% \overset{\text{"of"}}{*} \text{Original price}$$

$$\text{Final price} = \frac{60}{100} * \frac{40}{100} * \text{Original price}$$

$$\text{Final} = \frac{60}{100} * \frac{40}{100} * \$20$$

$$\frac{60 * 40 * 20}{100 * 100} \rightarrow \frac{6 * 4 * 2}{10} \rightarrow \frac{48}{10} \rightarrow \boxed{\$4.80}$$



Practice Problem Set #2

1) 105 is 15% of what number?

2) 30% of 80% of 1600 = ?

3) UM-Dearborn accepts about 65% of students who apply. At most universities, about 30% of accepted students enroll at that institution. If 10,000 students applied to UM-Dearborn in the last admissions cycle, approximately how many of those students ended up enrolling here?

Practice Problem Set #2 Solutions

1) 700

2) 384

3) 1,950 (or about 2,000) of those students enrolled

Proportions

- A proportion is just a statement that two ratios (fractions) are equal to one another:

$$\frac{2}{7} = \frac{8}{28}$$

- If we have a proportion where we don't know one value, we can use algebra to find it:

$$\frac{2}{6} = \frac{3}{y}$$

$$y * \frac{2}{6} = \frac{3}{y} * y \quad \rightarrow \quad y * \frac{2}{6} = 3 \quad \rightarrow \quad \frac{y * \frac{2}{6}}{\frac{2}{6}} = \frac{3}{\frac{2}{6}} \quad \rightarrow$$

$$y = 3 * \frac{6}{2} \quad \rightarrow \quad y = 9$$

Solving Problems with Proportions

Proportions are useful in everyday situations:

You want to drive to East Lansing to see a football game. The round trip is about 170 miles, and your car gets about 30 miles per gallon of gas. How much gas will you use for this round trip?

$$\frac{30 \text{ miles}}{1 \text{ gallon}} = \frac{170 \text{ miles}}{?}$$

$$? * \frac{30 \text{ miles}}{1 \text{ gallon}} = \frac{170 \text{ miles}}{?} * ?$$

$$\frac{? * \frac{30 \text{ miles}}{1 \text{ gallon}}}{\frac{30 \text{ miles}}{1 \text{ gallon}}} = \frac{170 \text{ miles}}{\frac{30 \text{ miles}}{1 \text{ gallon}}}$$

$$? = 170 \text{ miles} * \frac{1 \text{ gallon}}{30 \text{ miles}} \rightarrow ? = \frac{170}{30} \text{ gallons} \rightarrow \boxed{? = 5\frac{2}{3} \text{ gallons}}$$

Practice Problem Set #3

Set up proportions to solve the following problems:

1. Sam can buy 8 erasers for $5. How much would she pay for 14 erasers?

2. A certain type of paint is advertised as covering 450 ft^2 per gallon. How many one-gallon cans would be required to paint a room with a total wall space of 1200 ft^2?

Practice Problem Set #3 Solutions

1. $8.75

2. You'll use $2\frac{2}{3}$ gallons, so you'll need to buy 3 cans.