



Vectors I

Partially based on a vector tutorial prepared by Larry Friesen and Anne Gillis at Butler Community College



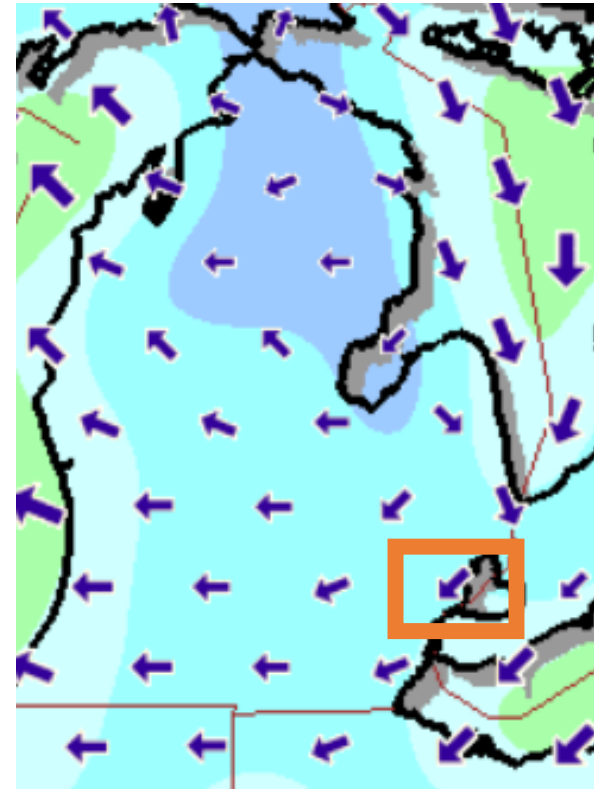
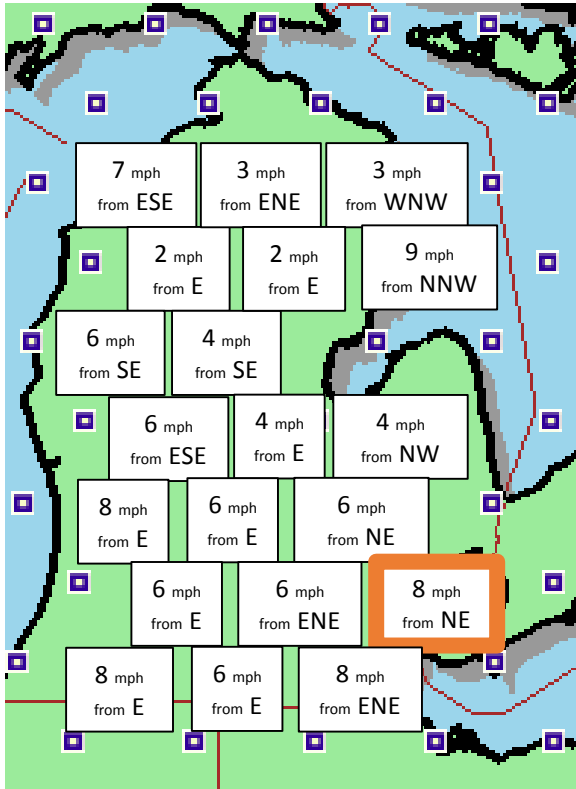
SCIENCE LEARNING CENTER

Introduction to Vectors

Why Bother With Vectors?

Imagine you wanted to represent the wind speed and direction across Michigan. (e.g., In Dearborn, the wind has a speed of 8 miles per hour, and is blowing from the northeast.)

Which of these figures makes it easier to see the overall wind pattern?



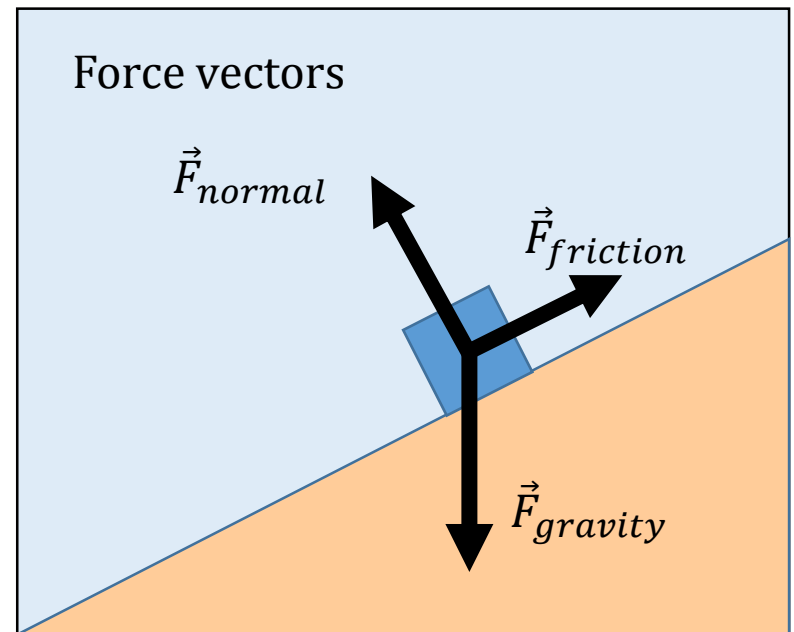
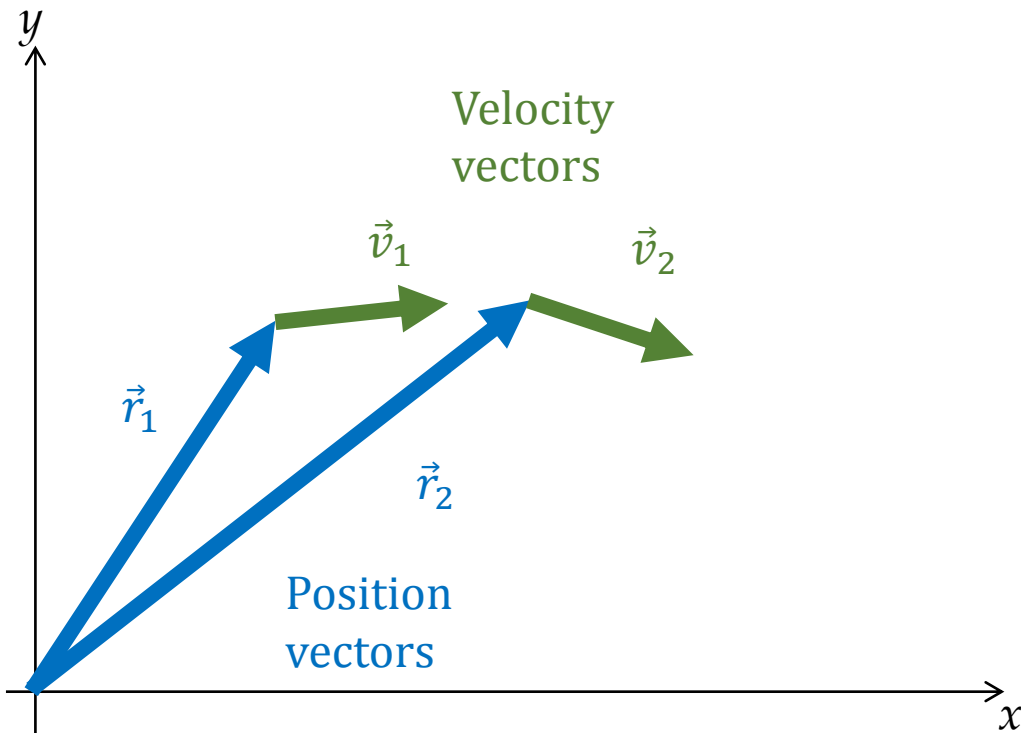
Why Bother With Vectors?

- Using vectors, we can represent the **size/amount** AND **direction** of the wind efficiently.
- Things that don't have a direction, like temperature or mass, are called **scalar** quantities. They wouldn't be represented with vectors.
- What things that have **both a size and a direction** can we represent with vectors?

Vectors in Physics

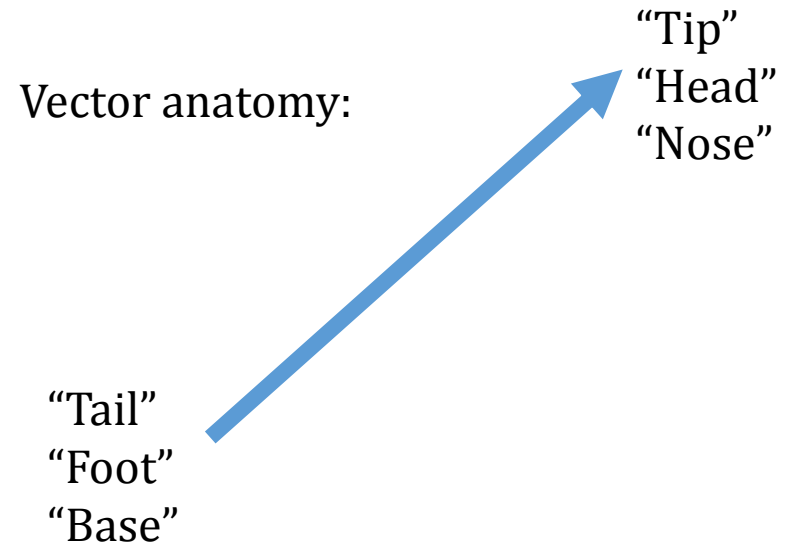
In physics, we use vectors to represent things like:

- **Position:**
How far away and in which direction from the origin is an object located?
- **Velocity:**
How fast and in which direction is it moving?
- **Force:**
How strong and in which direction is the push or pull on the object?

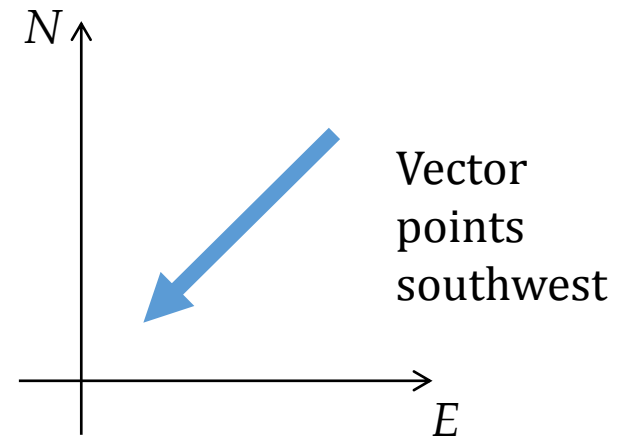
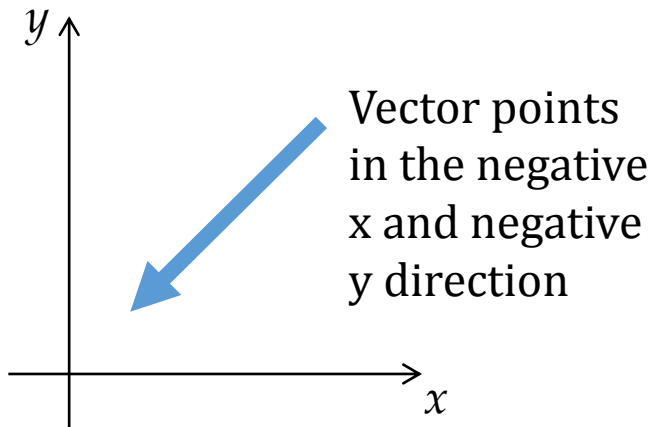


Representing Vectors

- There are many ways to represent a vector. The simplest way is with an arrow.



- We use coordinate systems to describe the direction of the arrow:



Components of Vectors

Mathematical Representations

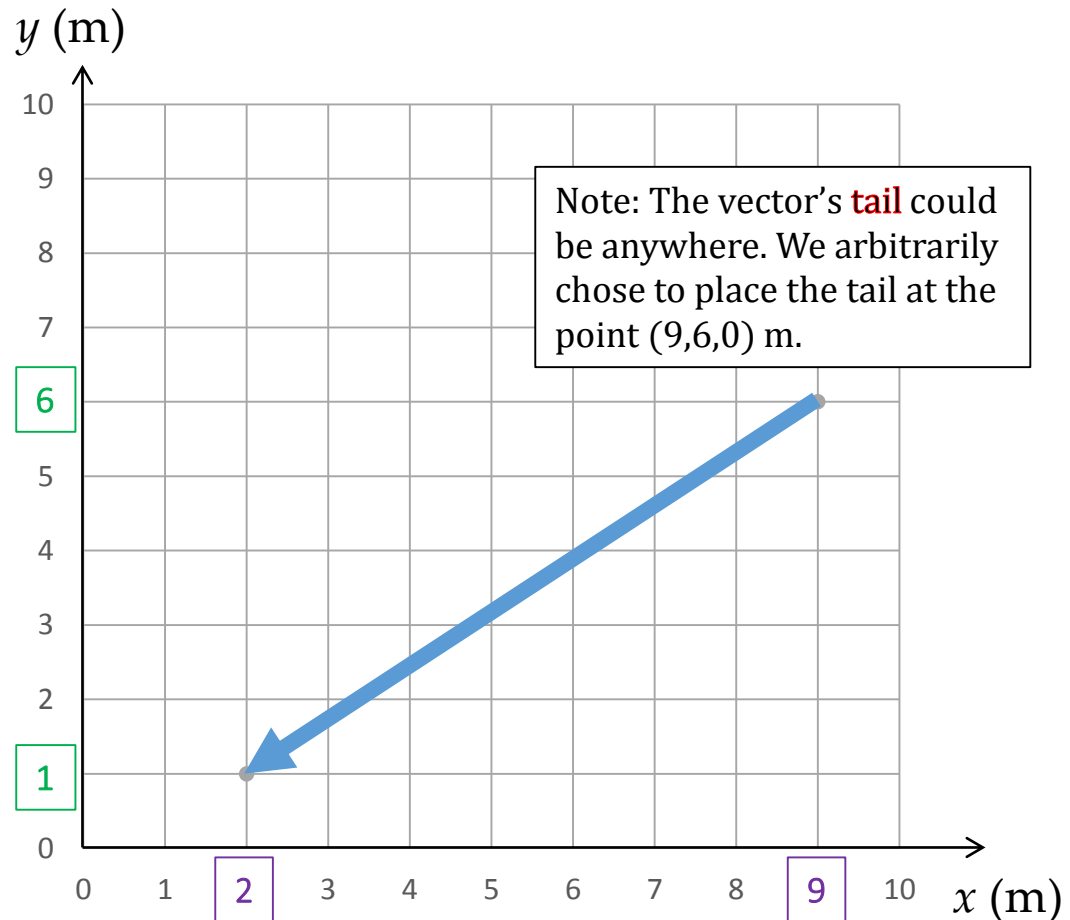
The image at right shows the three-dimensional vector:

$$\langle -7, -5, 0 \rangle \text{ m}$$

This notation means:

Starting from the “tail”,

- go 7 meters to the left ,
- go 5 meters down, and
- go 0 meters out of the page.



Components

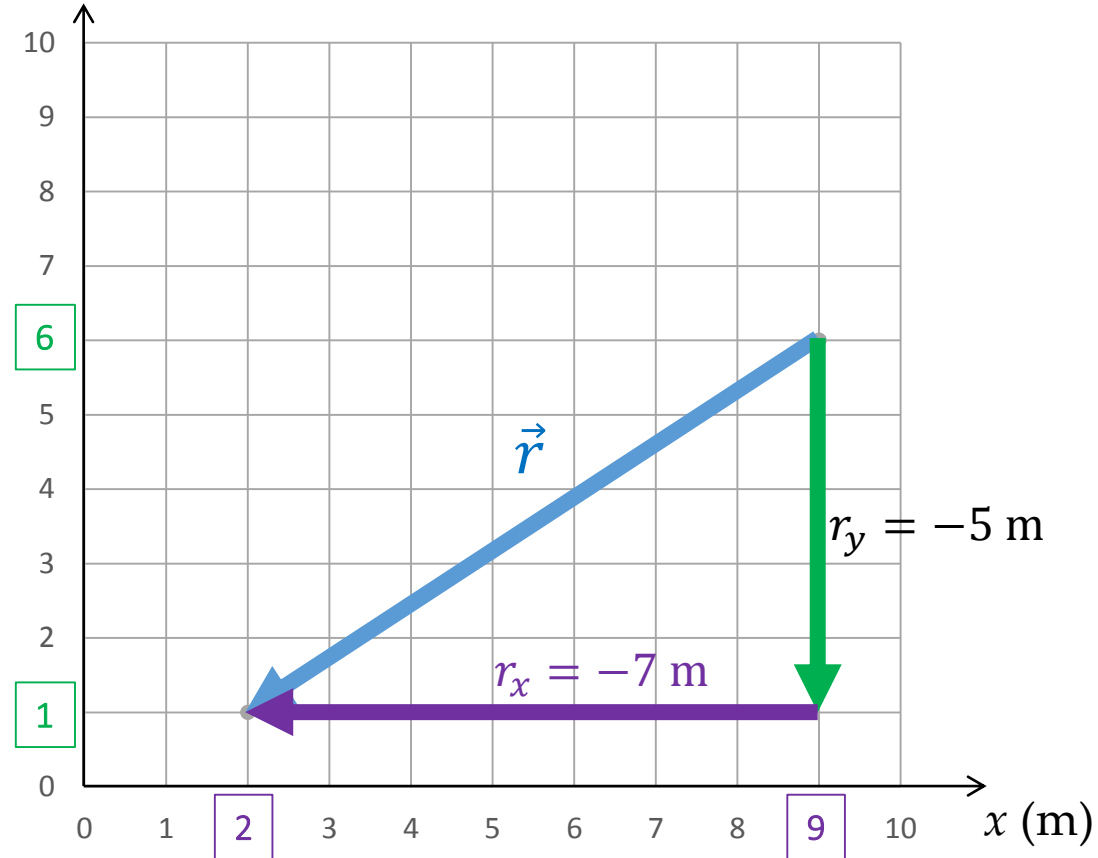
This is the overall vector

$$\vec{r} = \langle -7, -5, 0 \rangle \text{ m}$$

This is the x-component:
 $r_x = -7 \text{ m}$

This is the y-component:
 $r_y = -5 \text{ m}$

This is the z-component:
 $r_z = 0 \text{ m}$

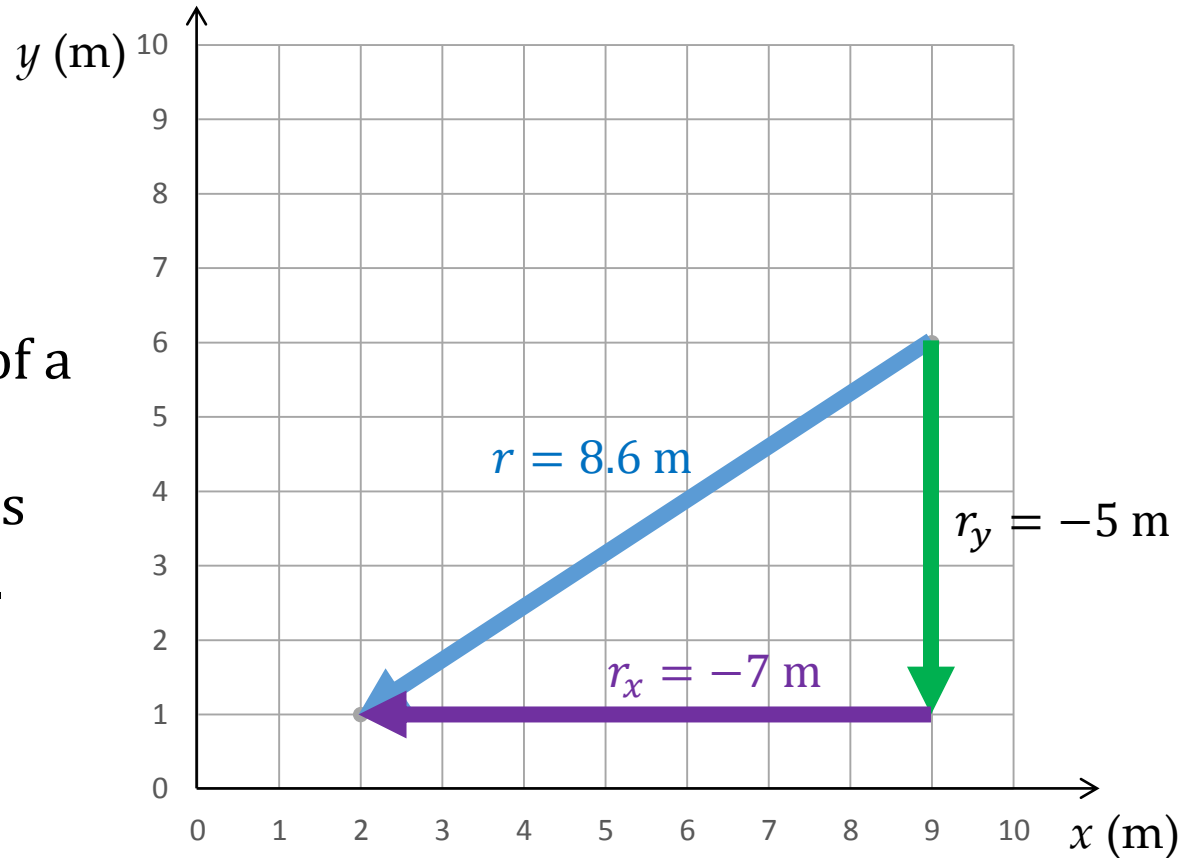


Note! We can use **any letter** to represent the overall vector

Magnitude

$$\vec{r} = \langle -7, -5, 0 \rangle \text{ m}$$

- The total size (length) of a vector \vec{r} is called its **magnitude**. Magnitude is represented as $|\vec{r}|$ or r .
- If you know a vector's components, you can calculate the vector's **magnitude** using the Pythagorean theorem:



$$|\vec{r}| = \sqrt{r_x^2 + r_y^2 + r_z^2}$$

$$|\vec{r}| = \sqrt{(-7 \text{ m})^2 + (-5 \text{ m})^2 + (0 \text{ m})^2}$$

$$|\vec{r}| = \sqrt{74} \text{ m}$$

$$|\vec{r}| \cong 8.6 \text{ m}$$

The **magnitude** is approximately **8.6 m**

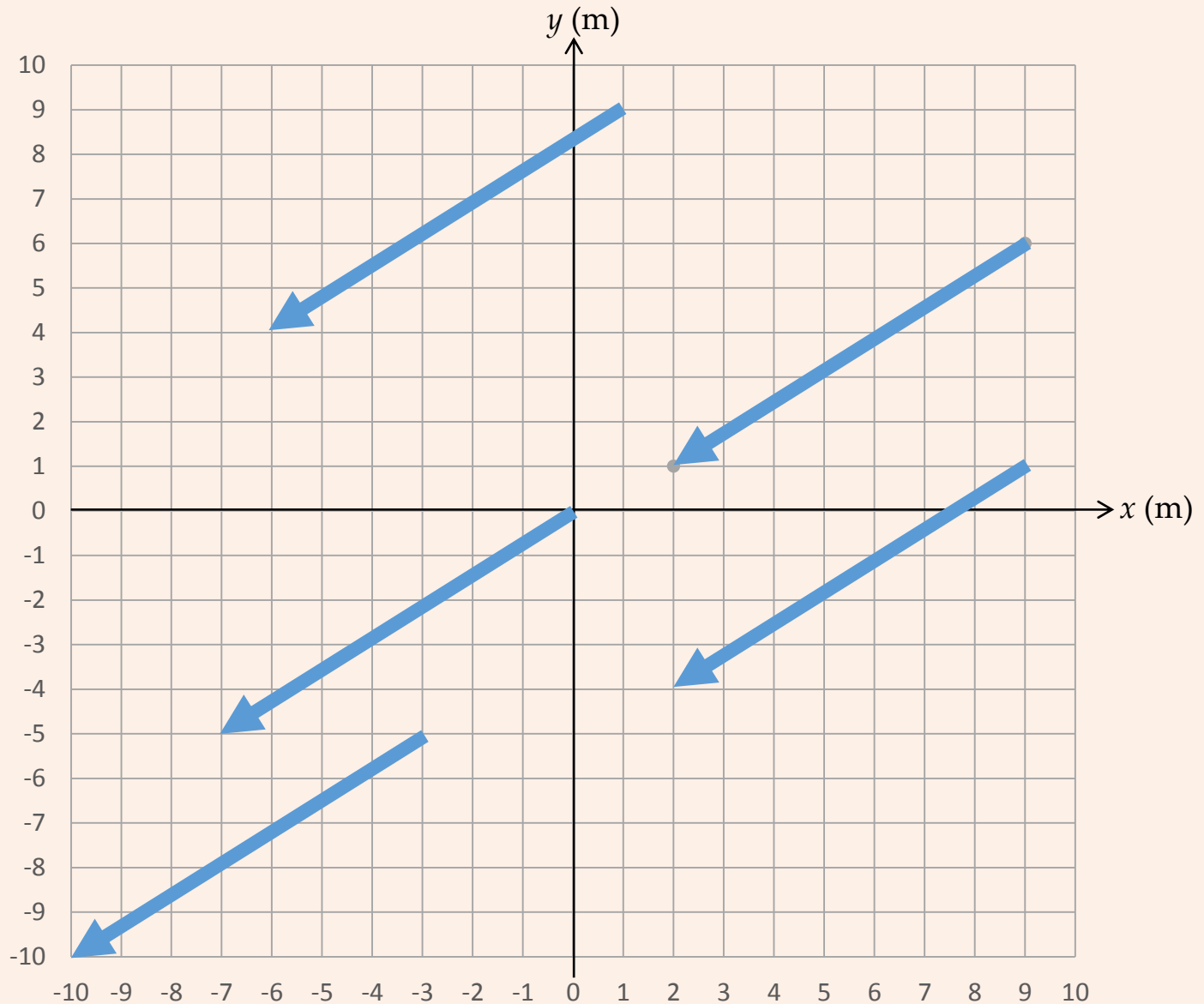
Equivalent Vectors

Test yourself:

Which of these
is the vector

$$\langle -7, -5, 0 \rangle \text{ m?}$$

(Assume all
z-components
are zero)



Equivalent Vectors

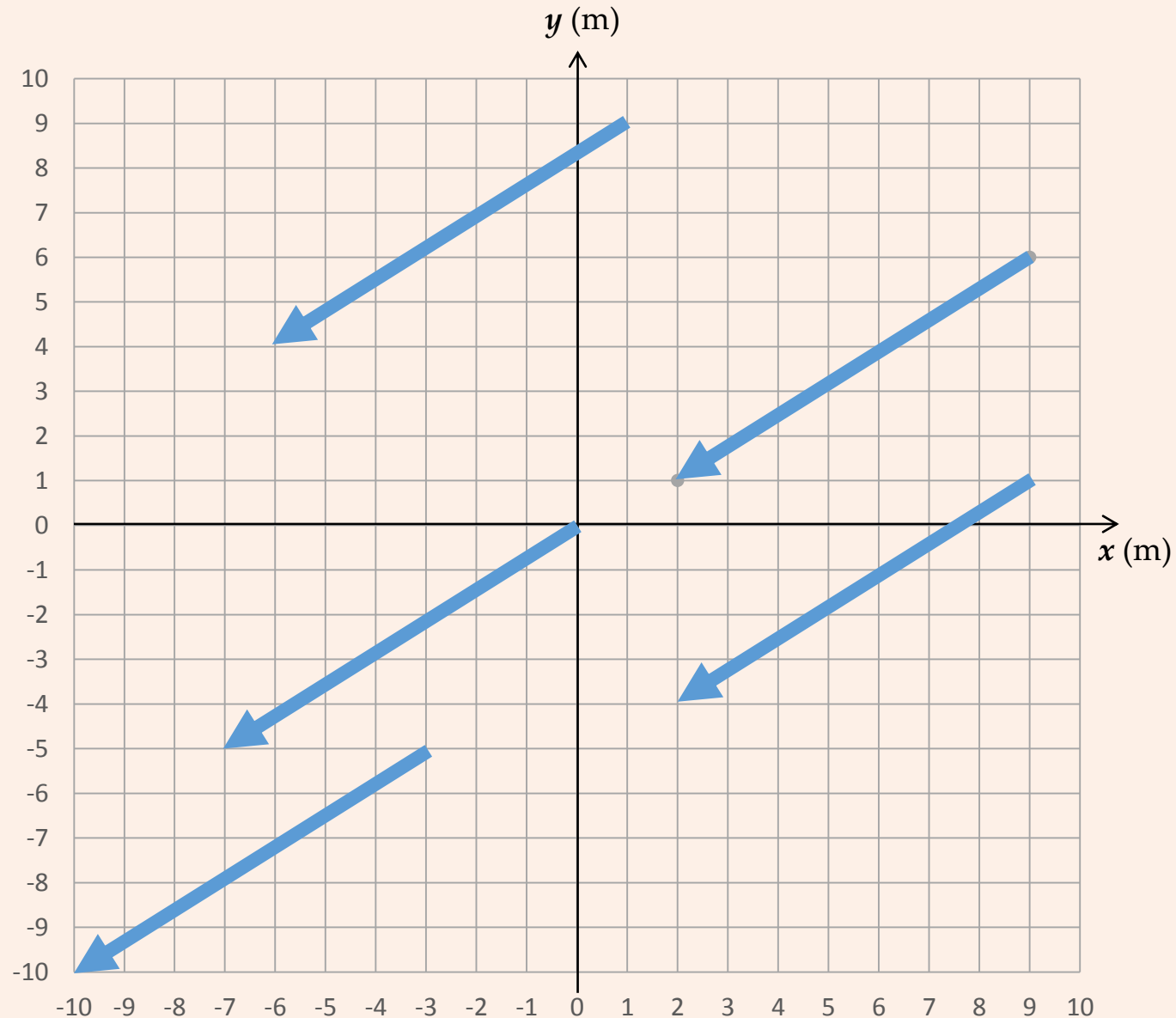
Which of these
is the vector

$\langle -7, -5, 0 \rangle$ m?

ALL of them!

You can move
vectors around.

As long as they
have the **same**
magnitude AND
direction, they
are **equivalent**.

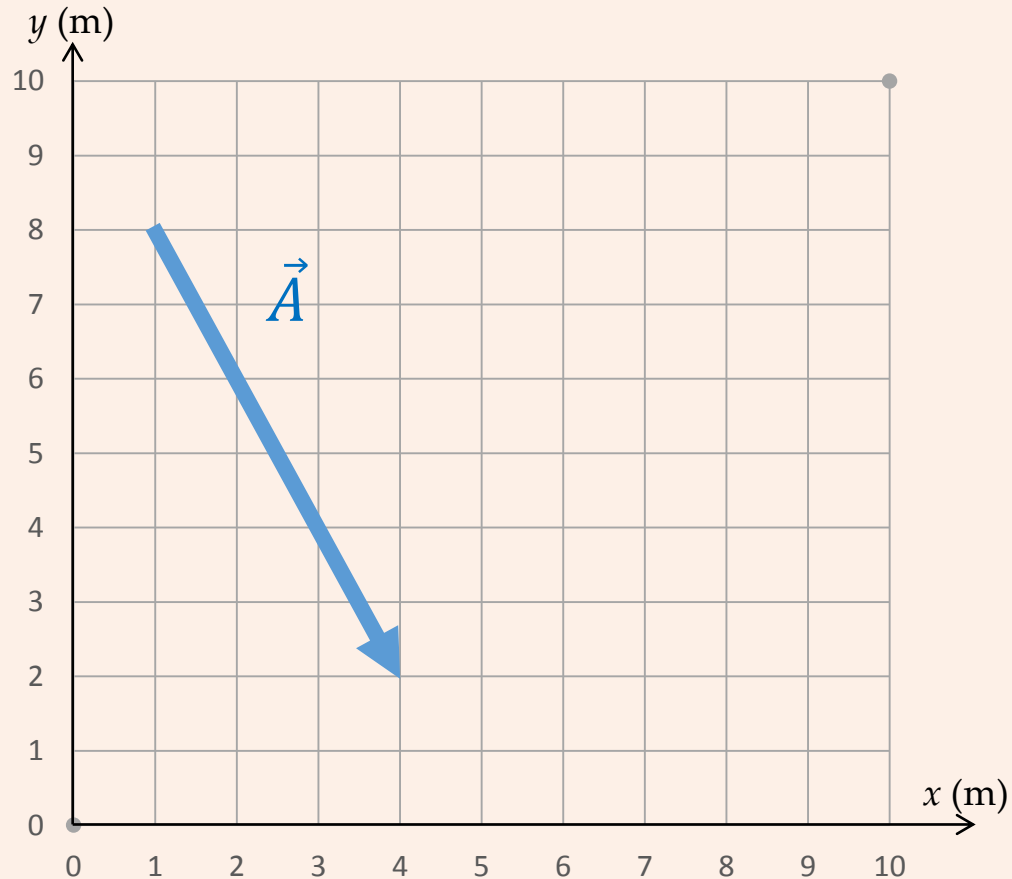


Practice #1

Q1. Find the x- and y-components of the vector \vec{A} at right. (You can assume the z-component is zero.)

Q2. Sketch the vector:
 $\vec{B} = \langle -3, 2, 0 \rangle$ m.

Q3. $\vec{B} = \langle -3, 2, 0 \rangle$ m. Calculate the **magnitude** of \vec{B} .



See end of packet for solutions

Unit Vector Notation

Mathematical Representations: Unit Vectors

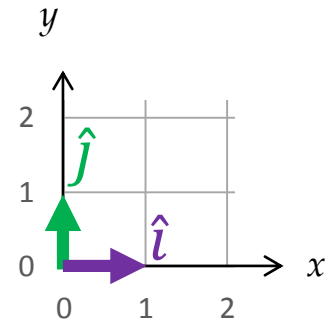
We've been representing vectors as $\langle A_x, A_y, A_z \rangle$.

We can also represent vectors mathematically using “unit vectors”.

- Unit vectors are defined to have a magnitude of 1
- Unit vectors are dimensionless (they don't have units)

We often use the unit vectors that point in the $+x$, $+y$, and $+z$ directions:

- \hat{i} (or \hat{x}) is the unit vector that points in the $+x$ direction (pronounce \hat{i} as “eye-hat”)
- \hat{j} (or \hat{y}) is the unit vector that points in the $+y$ direction
- \hat{k} (or \hat{z}) is the unit vector that points in the $+z$ direction



Note: \hat{i} , \hat{j} and \hat{k} notation is commonly used in engineering courses

Mathematical Representations: Unit Vectors

\hat{i} (or \hat{x}) means “one unit in the $+x$ direction”

\hat{j} (or \hat{y}) means “one unit in the $+y$ direction”

\hat{k} (or \hat{z}) means “one unit in the $+z$ direction”

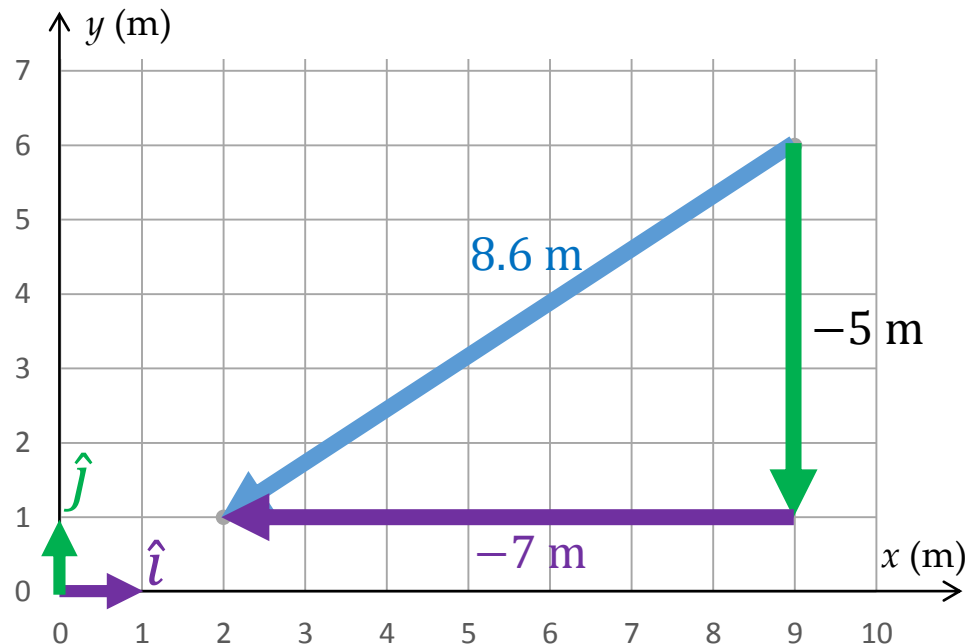
So we can write the vector at right as:

$$\langle -7, -5, 0 \rangle \text{ m}$$

OR, using unit vectors, as:

$$(-7\hat{i} - 5\hat{j} + 0\hat{k}) \text{ m}$$

(Note that we have to keep track of each **direction separately**. We cannot combine the 7, 5, and 0)

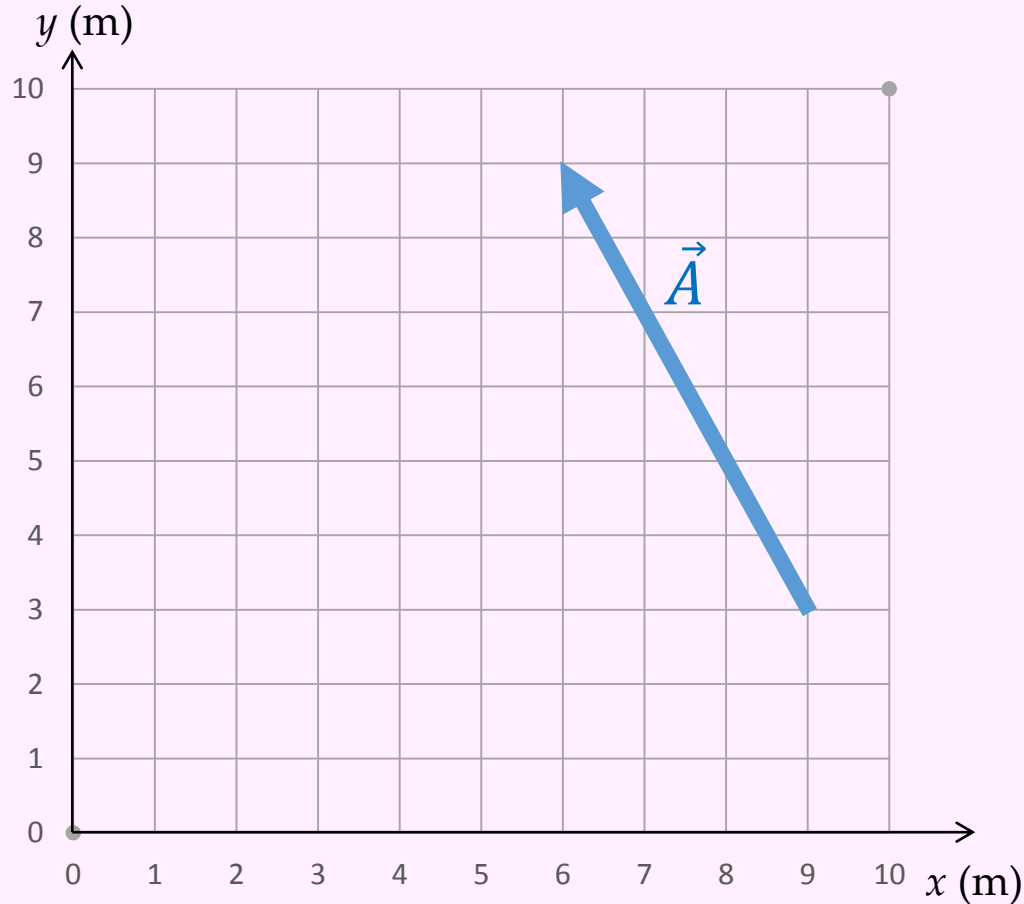


Practice #2

Q1. Write the vector \vec{A} (shown at right), using \hat{i} , \hat{j} , \hat{k} unit vector notation:

Q2. Given $\vec{B} = (2\hat{i} - 3\hat{j} + 0\hat{k})$ m, sketch \vec{B} :

See end of packet for solutions



Unit Vectors

Unit vectors can point in any direction.

To find a unit vector \hat{r} in the direction of any vector \vec{r} , divide the vector \vec{r} by its magnitude $|\vec{r}|$:

$$\hat{r} = \frac{\vec{r}}{|\vec{r}|}$$

For the example at right:

$$\hat{r} = \frac{-7, -5, 0 \text{ m}}{8.6 \text{ m}}$$

Note that the units cancel out. \hat{r} is dimensionless.

$$\hat{r} = \langle -0.814, -0.581, 0 \rangle$$

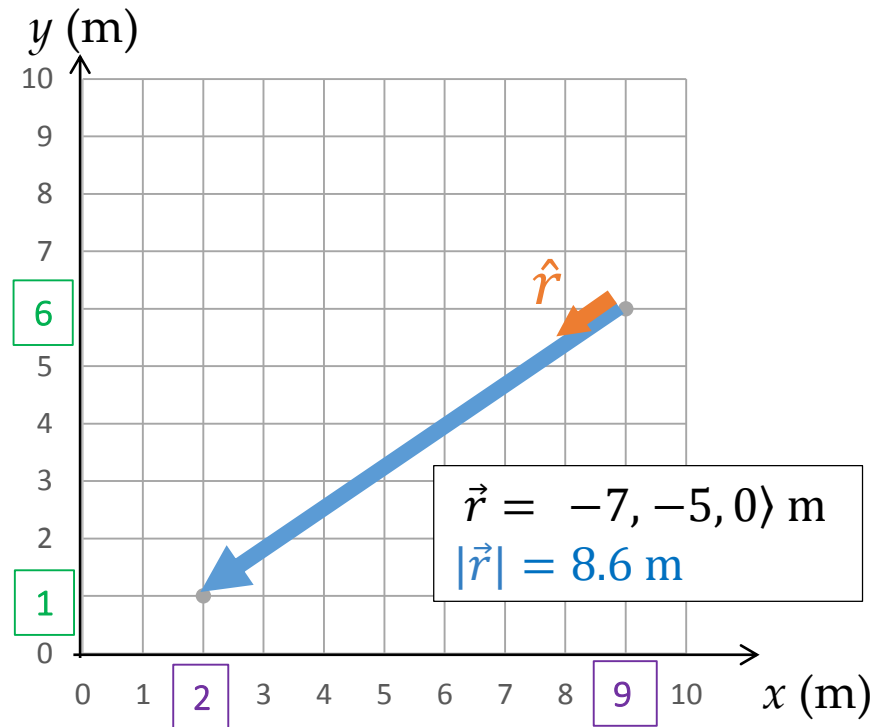
Check:

$$|\hat{r}| = \sqrt{(-0.814)^2 + (-0.581)^2 + (0)^2}$$

$$|\hat{r}| = \sqrt{1.00}$$

$$|\hat{r}| = 1 \quad \checkmark$$

\hat{r} is a unit vector that points in the direction of \vec{r}



Practice #3

Q1. Is $\vec{c} = \left\langle -\frac{\sqrt{3}}{2}, -\frac{1}{2}, 0 \right\rangle$ a unit vector?

Q2. Is $\vec{d} = (1\hat{i} + 1\hat{j})$ a unit vector?

Q3. Find the unit vector in the direction of $\vec{a} = \left\langle \frac{3}{2}, -1, 0 \right\rangle$ m.

See end of packet for solutions

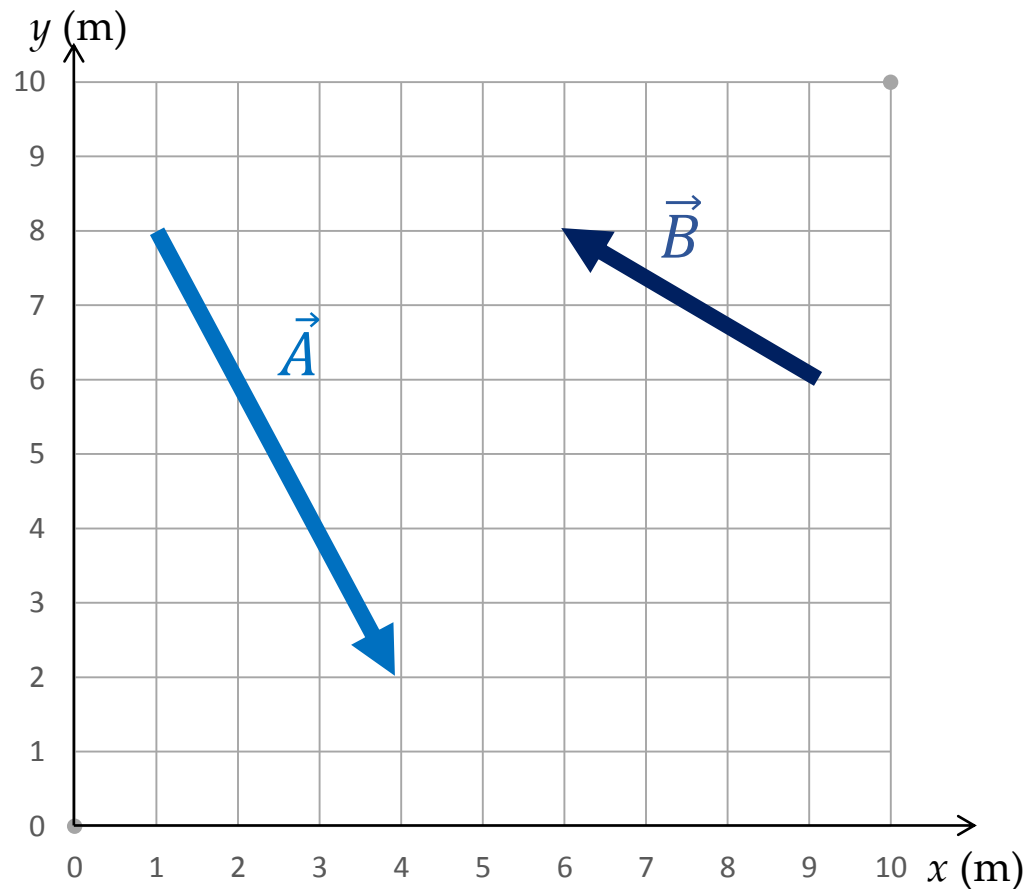
Adding Vectors

Adding Vectors Graphically

Vectors can be combined following specific rules for addition, multiplication, etc.

In the next few slides, we'll describe how to add the vectors \vec{A} and \vec{B} at right.

- Remember we can move vectors around, as long as we don't change their magnitude or direction!

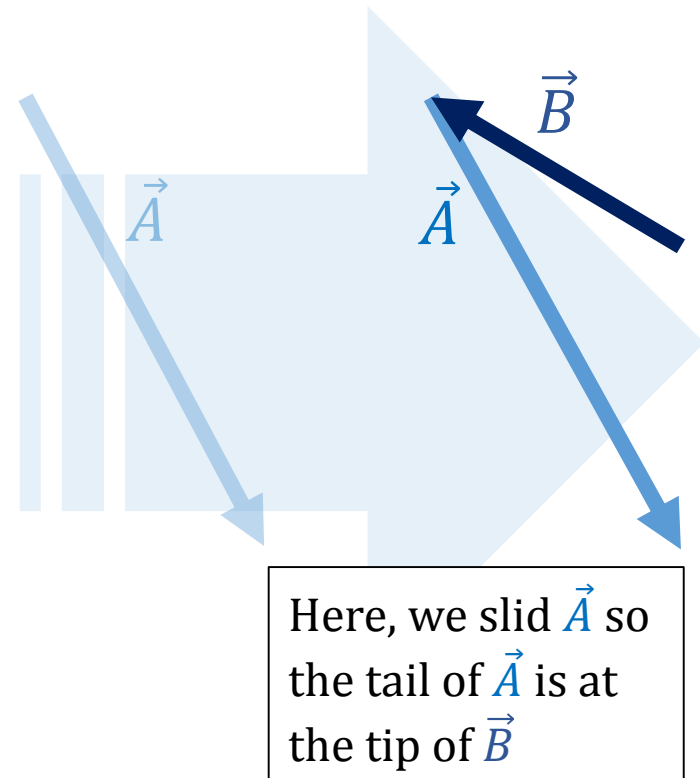
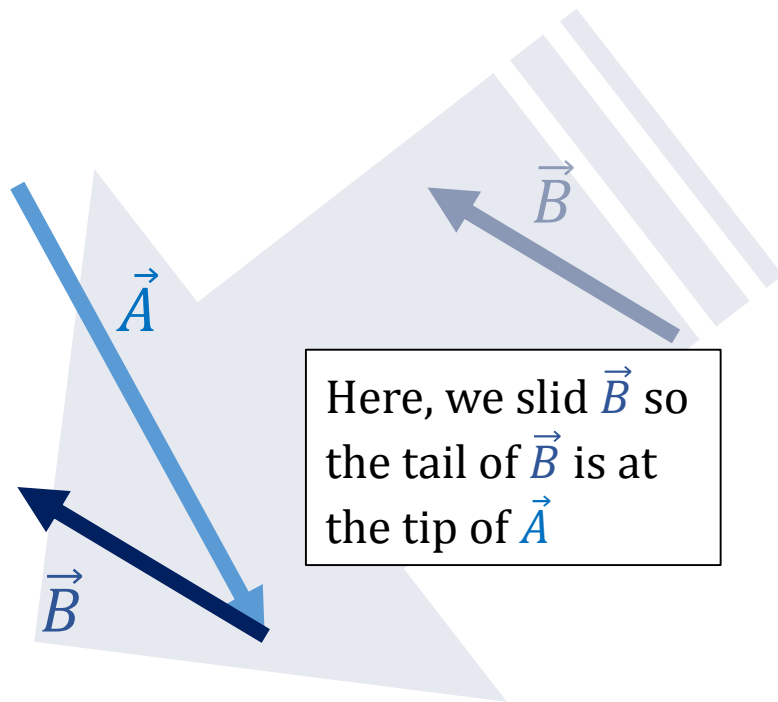


Adding Vectors Graphically

Step 1: Move one (or more) of the vectors to arrange them “tail to tip” or “tip to tail”

- It doesn't matter which of the vectors we move...

$$\vec{A} + \vec{B} \text{ is equivalent to } \vec{B} + \vec{A}$$



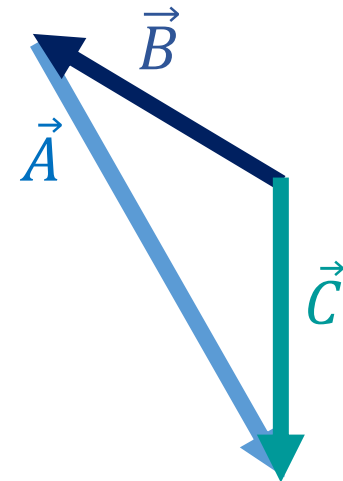
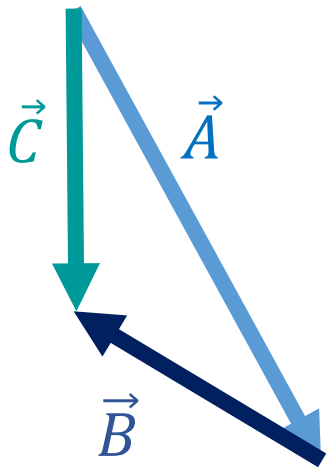
Adding Vectors Graphically

Step 2: Draw vector $\vec{C} = \vec{A} + \vec{B}$:

After arranging \vec{A} and \vec{B} tip to tail, draw vector \vec{C} . Start from the first **tail** in your “tip-to-tail” chain of vectors, and end at the last **tip**.

! **Caution:** This means the resulting vector ends up tip-to-tip, **NOT** tip-to-tail.

$$\vec{A} + \vec{B} = \vec{C} \quad \text{is equivalent to} \quad \vec{B} + \vec{A} = \vec{C}$$

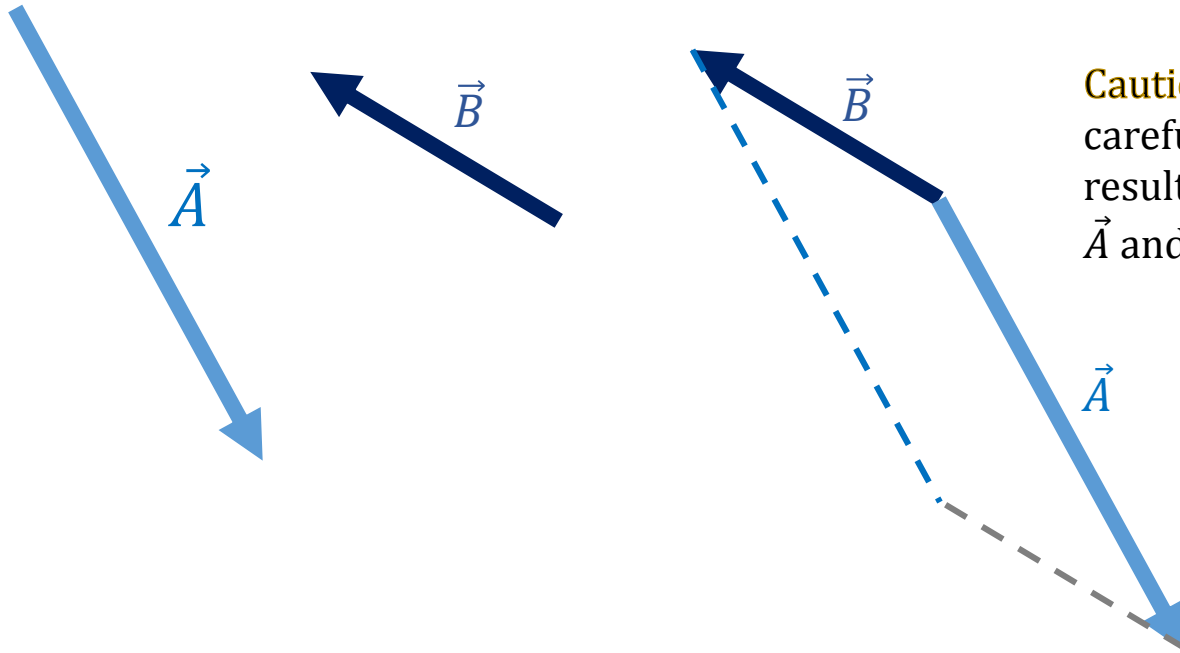


The resulting vector \vec{C} is equivalent!

Alternate Method of Addition

You can also use a parallelogram to add two vectors graphically.

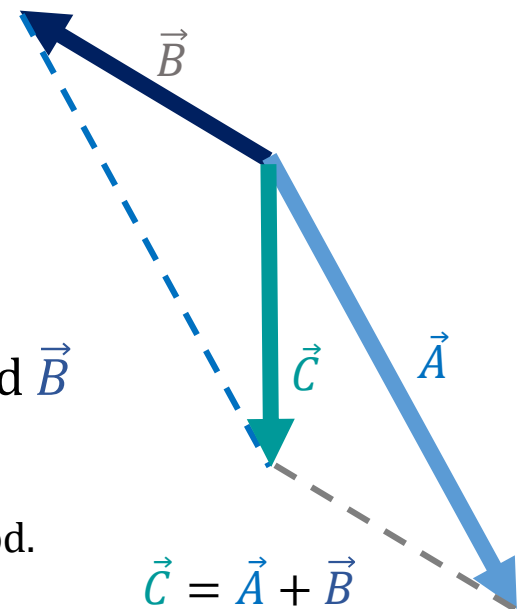
Step 1: Arrange vectors \vec{A} and \vec{B} tail to tail, and carefully sketch a parallelogram.



Caution: Draw the parallelogram carefully! Don't just draw the resultant vector randomly between \vec{A} and \vec{B} without following Step 1!

Step 2: Draw the resultant \vec{C} vector from the tails of \vec{A} and \vec{B} to the opposite corner of the parallelogram.

Note that this gives the same resultant vector as the previous method.



Adding Vectors Using Components

Graphical vector addition doesn't require a coordinate system.

If we have a coordinate system and we know the components of each vector, we just add each component independently.

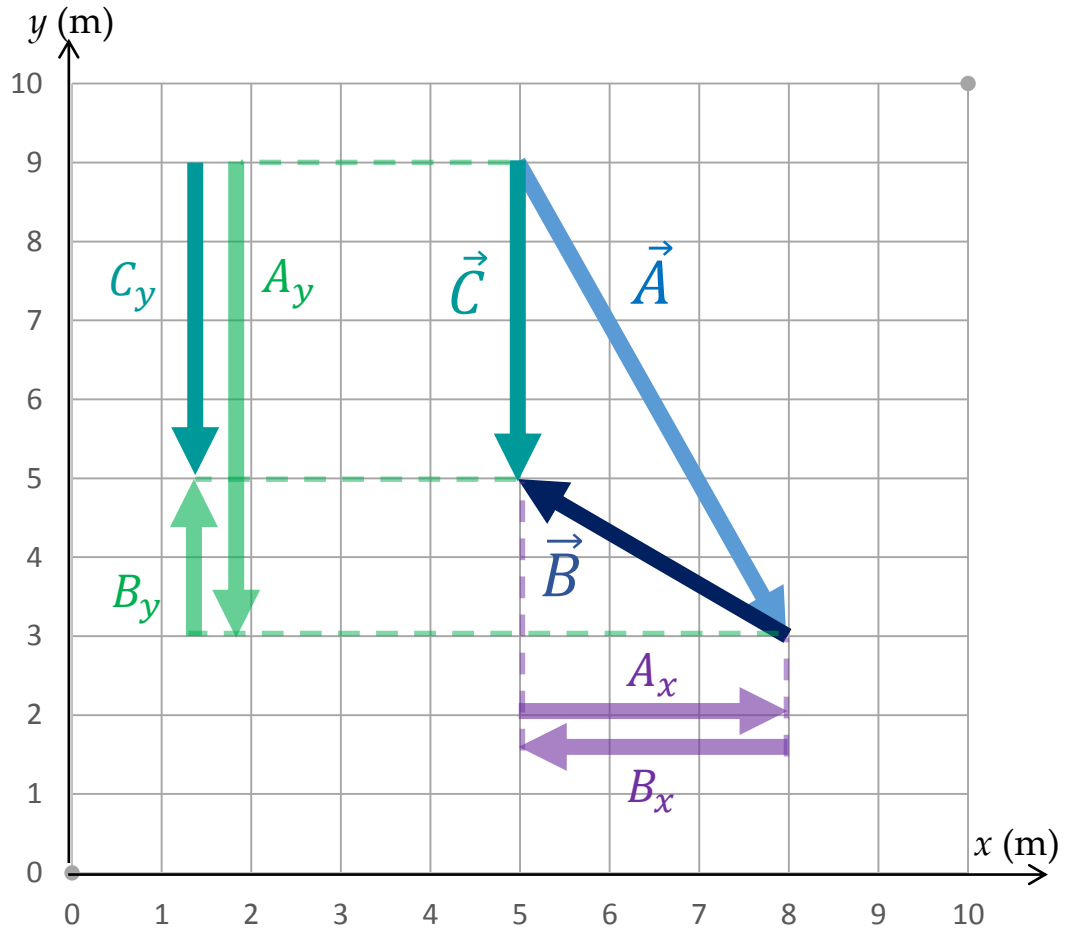
Given: $\vec{A} = (3\hat{i} - 6\hat{j} + 0\hat{k}) \text{ m}$
 $\vec{B} = (-3\hat{i} + 2\hat{j} + 0\hat{k}) \text{ m}$

We calculate $\vec{C} = \vec{A} + \vec{B}$:

$$\begin{array}{r} \vec{A} \\ + \vec{B} \\ \hline \vec{C} \end{array} \quad \begin{array}{l} (3\hat{i} - 6\hat{j} + 0\hat{k}) \text{ m} \\ + (-3\hat{i} + 2\hat{j} + 0\hat{k}) \text{ m} \\ \hline (0\hat{i} - 4\hat{j} + 0\hat{k}) \text{ m} \end{array}$$

$$C_x = A_x + B_x = 0 \text{ m}$$

$$C_y = A_y + B_y = -4 \text{ m}$$



$$\vec{C} = (0\hat{i} - 4\hat{j} + 0\hat{k}) \text{ m}$$

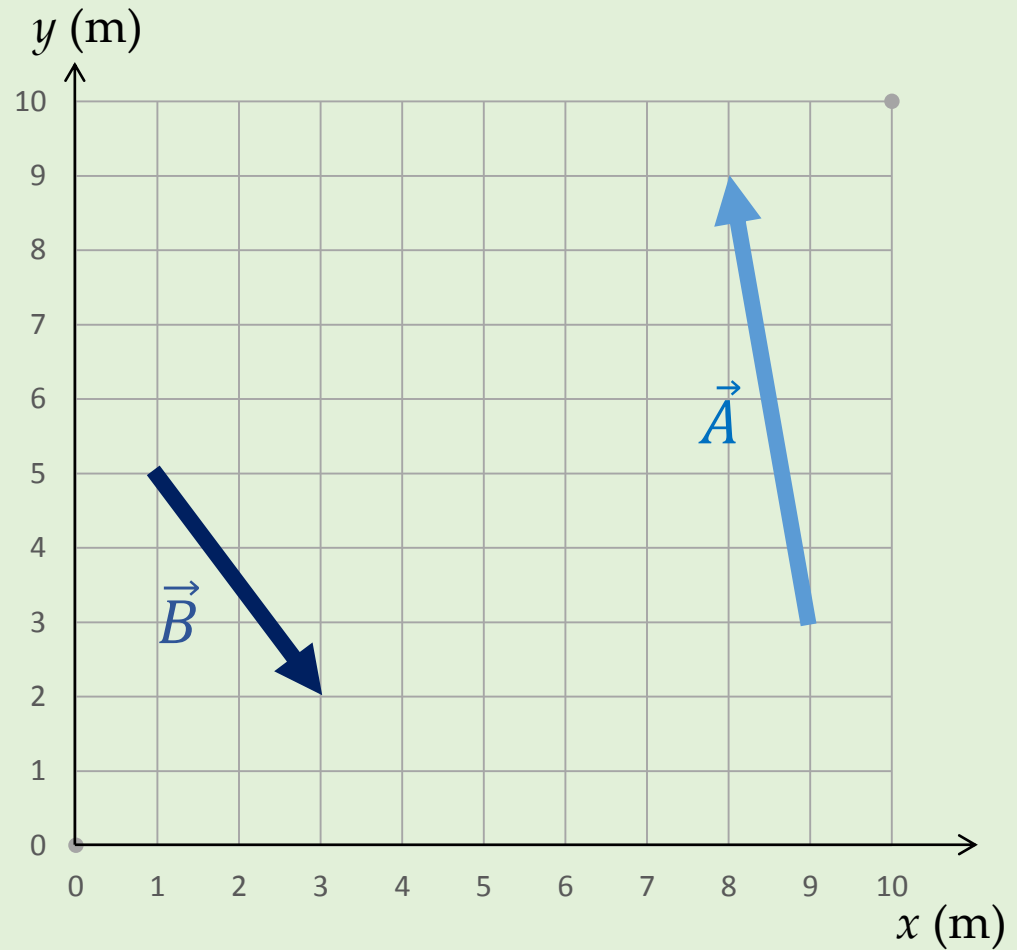
Note that this gives the same resultant vector as the previous methods.

Practice #4

Q1. Use **graphical** vector addition to draw the vector

$$\vec{C} = \vec{A} + \vec{B}$$

See end of packet for solutions



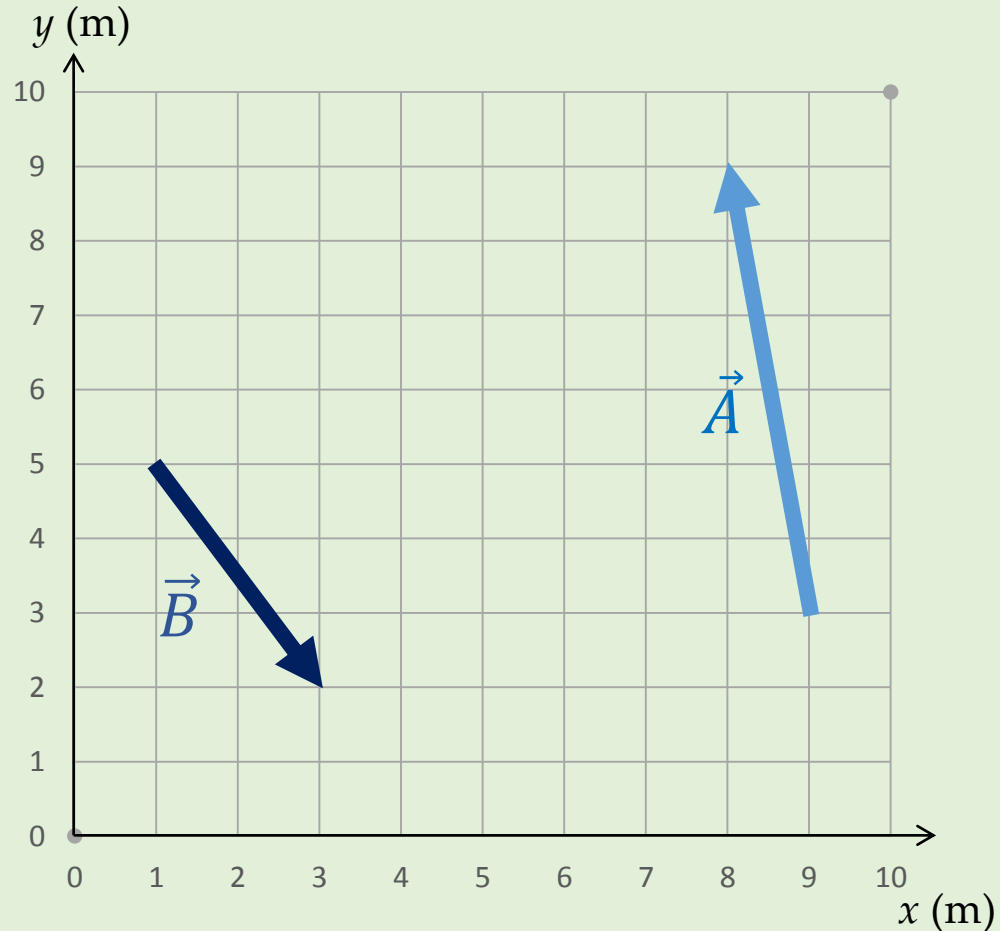
Practice #5

Q1. Write vectors \vec{A} and \vec{B} in unit-vector notation

Q2. Use vector addition **by components** to find the vector

$$\vec{C} = \vec{A} + \vec{B}$$

Q3. Does your answer match the graphical addition of these vectors in Practice #4?



See end of packet for solutions

Multiplying and Subtracting Vectors

Multiplying Vectors by Scalars

So far, we have only considered vector addition. Let's consider multiplication:

- You can multiply a vector by a scalar. This changes its magnitude.

$$\vec{A} = \langle 3, -6, 0 \rangle \text{ m}$$

$$\frac{1}{2}\vec{A} = \frac{1}{2}\langle 3, -6, 0 \rangle \text{ m}$$

$$\frac{1}{2}\vec{A} = \langle \frac{3}{2}, -3, 0 \rangle \text{ m}$$

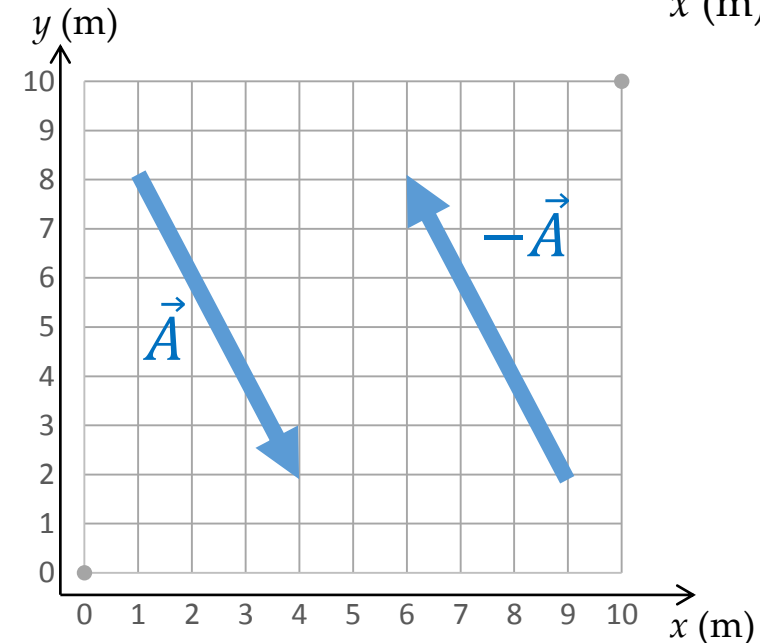
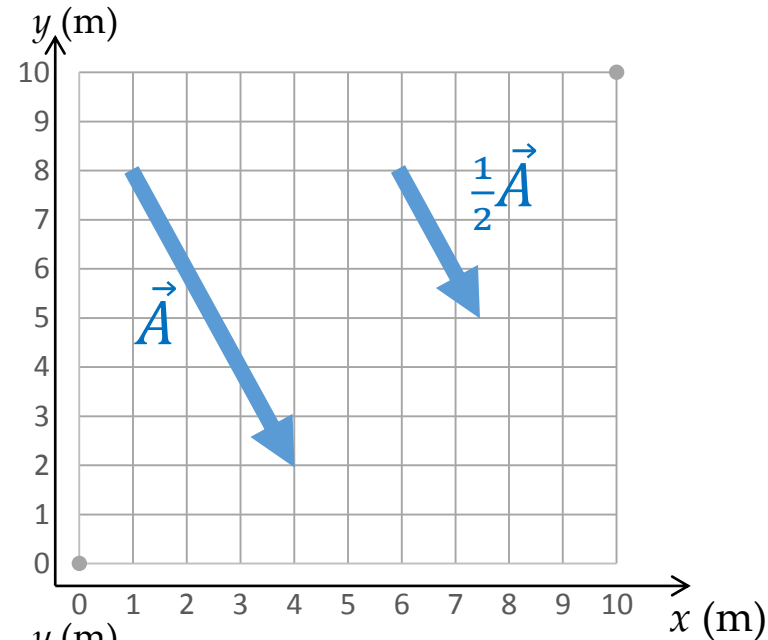
- Multiplying by a **negative** number **reverses the direction** of the vector.

$$\vec{A} = \langle 3, -6, 0 \rangle \text{ m}$$

$$-\vec{A} = -\langle 3, -6, 0 \rangle \text{ m}$$

$$-\vec{A} = \langle -3, 6, 0 \rangle \text{ m}$$

You can also multiply vectors by vectors! See the Vectors III module for details.

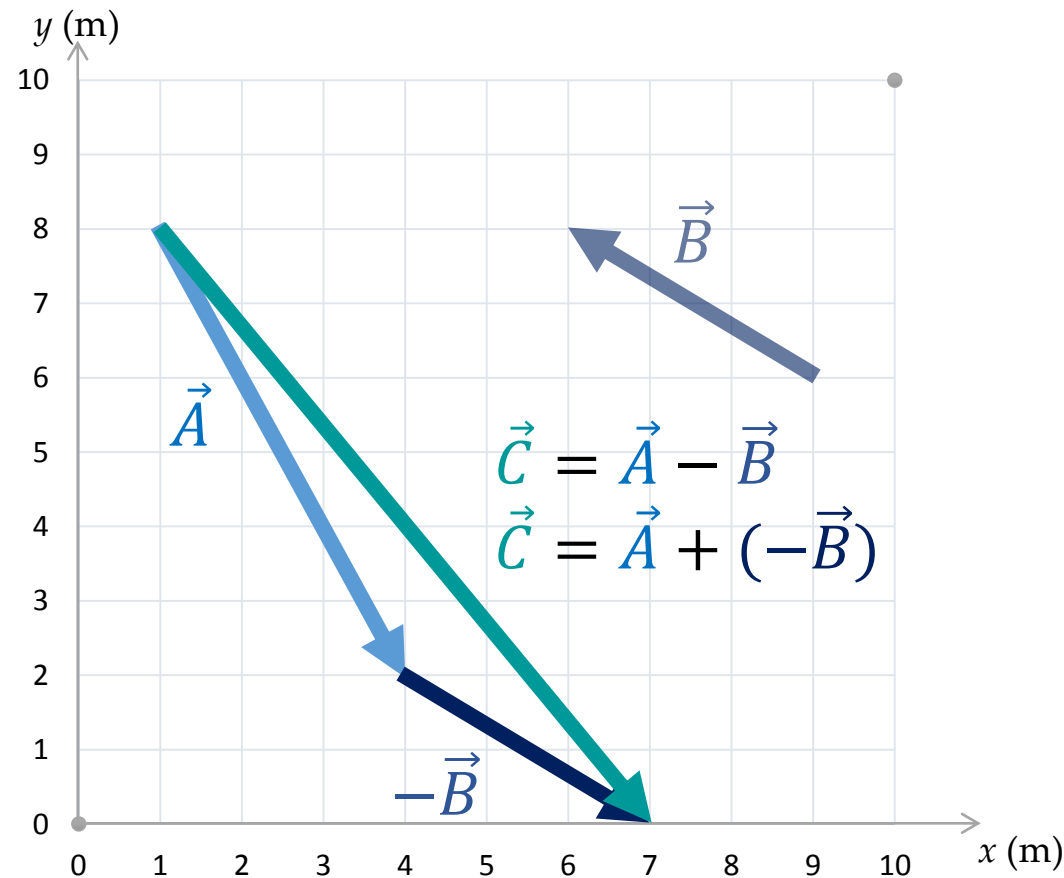


Subtracting Vectors

Now that we can multiply by a negative number, we can do **subtraction**:

$$\vec{A} - \vec{B} \text{ is just } \vec{A} + (-\vec{B})$$

So to graphically calculate $\vec{A} - \vec{B}$, we just reverse the direction of \vec{B} to get $-\vec{B}$, then add \vec{A} and $(-\vec{B})$ as usual, “tail to tip” or “tip to tail”.



This graphical subtraction matches what we get using components:

$$\vec{A} = \langle 3, -6, 0 \rangle \text{ m}$$

$$\vec{B} = \langle -3, 2, 0 \rangle \text{ m}$$

$$\begin{array}{r} \vec{A} \\ -\vec{B} \\ \hline \vec{C} \end{array} \quad \begin{array}{r} \langle 3, -6, 0 \rangle \text{ m} \\ - \langle -3, 2, 0 \rangle \text{ m} \\ \hline \langle 6, -8, 0 \rangle \text{ m} \end{array}$$

Alternate Method of Subtraction

You can also use a triangle to subtract two vectors graphically.

Step 1: The vectors \vec{A} and \vec{B} are arranged tail to tail in order to sketch a triangle.

Step 2: The final vector is drawn from the tip of the second vector (\vec{B}) to the tip of the first one (\vec{A}).

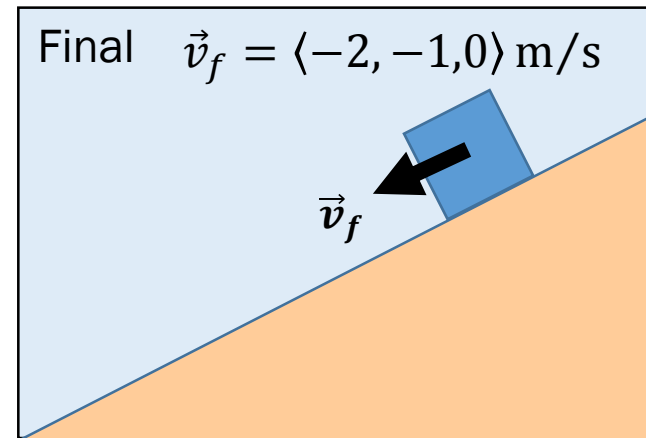
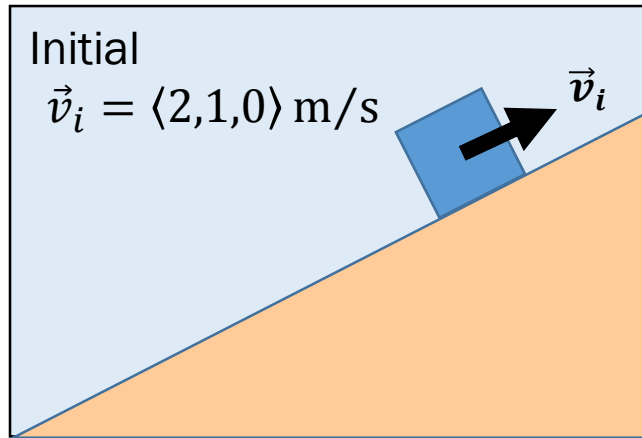


Note that this gives the same result as the previous method.

Applied Vector Subtraction

We define $\overline{\Delta v}$ as the change in the velocity. Mathematically, $\overline{\Delta v} = \overline{v_f} - \overline{v_i}$

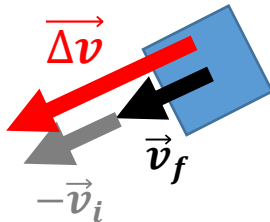
The initial and final velocity vectors are given in the figures:



We can find $\overline{\Delta v}$ **graphically** and **by components**:

Graphically:

$$\overline{\Delta v} = \overline{v_f} + (-\overline{v_i})$$



By components:

$$\overline{\Delta v} = \overline{v_f} - \overline{v_i}$$

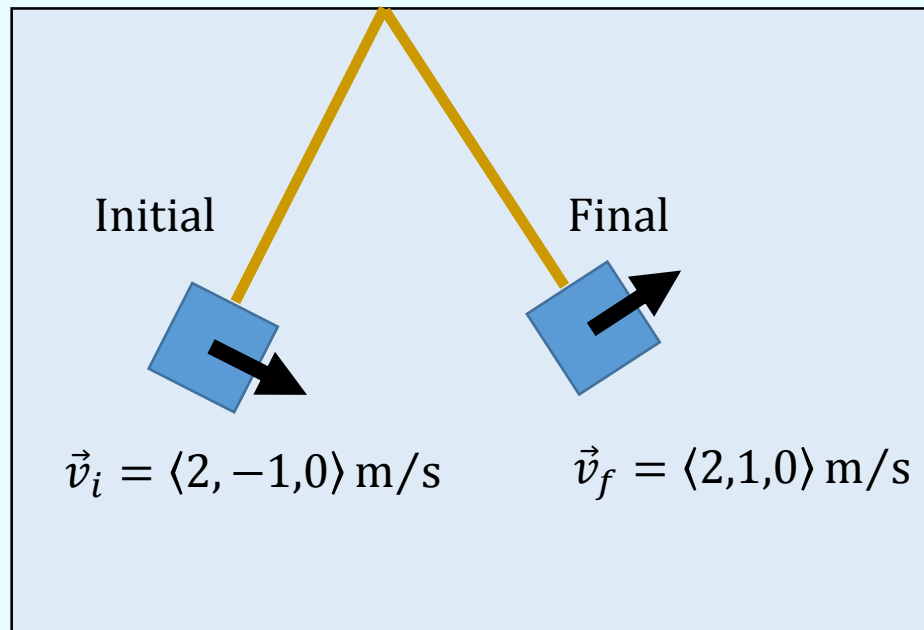
$\overline{v_f}$		$\langle -2, -1, 0 \rangle \text{ m/s}$
$-\overline{v_i}$	-	$\langle 2, 1, 0 \rangle \text{ m/s}$
$\overline{\Delta v}$		$\langle -4, -2, 0 \rangle \text{ m/s}$

Practice #6

Q1. The initial and final velocity vectors are given in the figure.

We define $\overline{\Delta v} = \overline{v_f} - \overline{v_i}$. For this scenario, find $\overline{\Delta v}$:

- i. graphically and
- ii. by components.



See end of packet for solutions

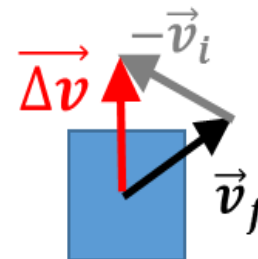
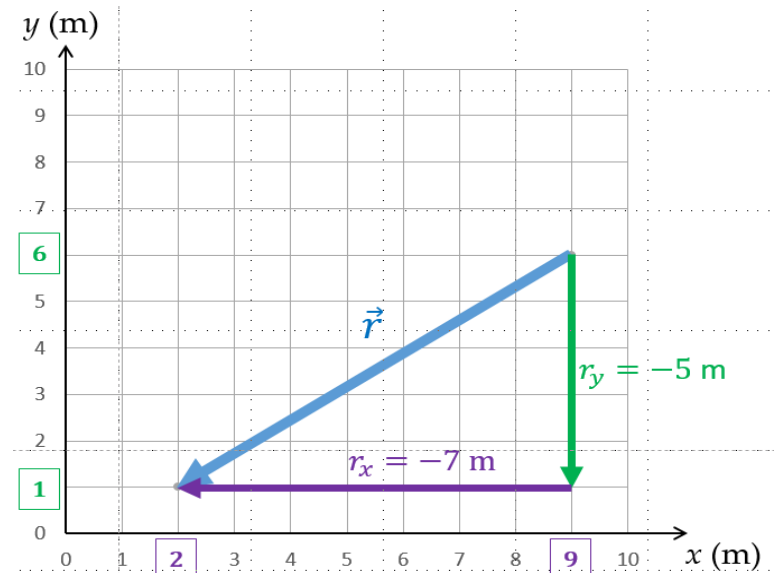
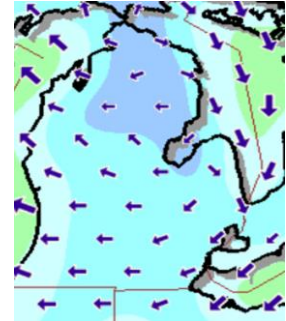
Conclusion

Recap

After working through this module, we expect that you can now:

- Distinguish vector quantities from scalar quantities
- Represent vectors graphically and mathematically (including using unit-vector notation)
- Use vector components to find the magnitude of a vector.
- Add and subtract vectors graphically and by using components
- Multiply vectors by scalars

If you still have questions about any of this, please ask an SLC assistant!



Take the Post Test!

- You have completed the tutorial portion of the Vectors 1 Module
- When you feel that you understand the principles outlined in this tutorial, get a post test from the Science Learning Center personnel.
- Complete the post test and turn it in to be checked.
- If correct, you will be able to move on to the Vectors 2 Module.
- If incorrect, you may review the module and retake the posttest as many times as is needed in order to pass.

Practice Problem Solutions

Practice #1 Solution

Q1. Find the x- and y- components of the vector \vec{A} at right. (You can assume the z-component is zero.)

$$\begin{aligned} A_x &= 3 \text{ m} \\ A_y &= -6 \text{ m} \\ A_z &= 0 \text{ m} \end{aligned}$$

$$\vec{A} = \langle 3, -6, 0 \rangle \text{ m}$$

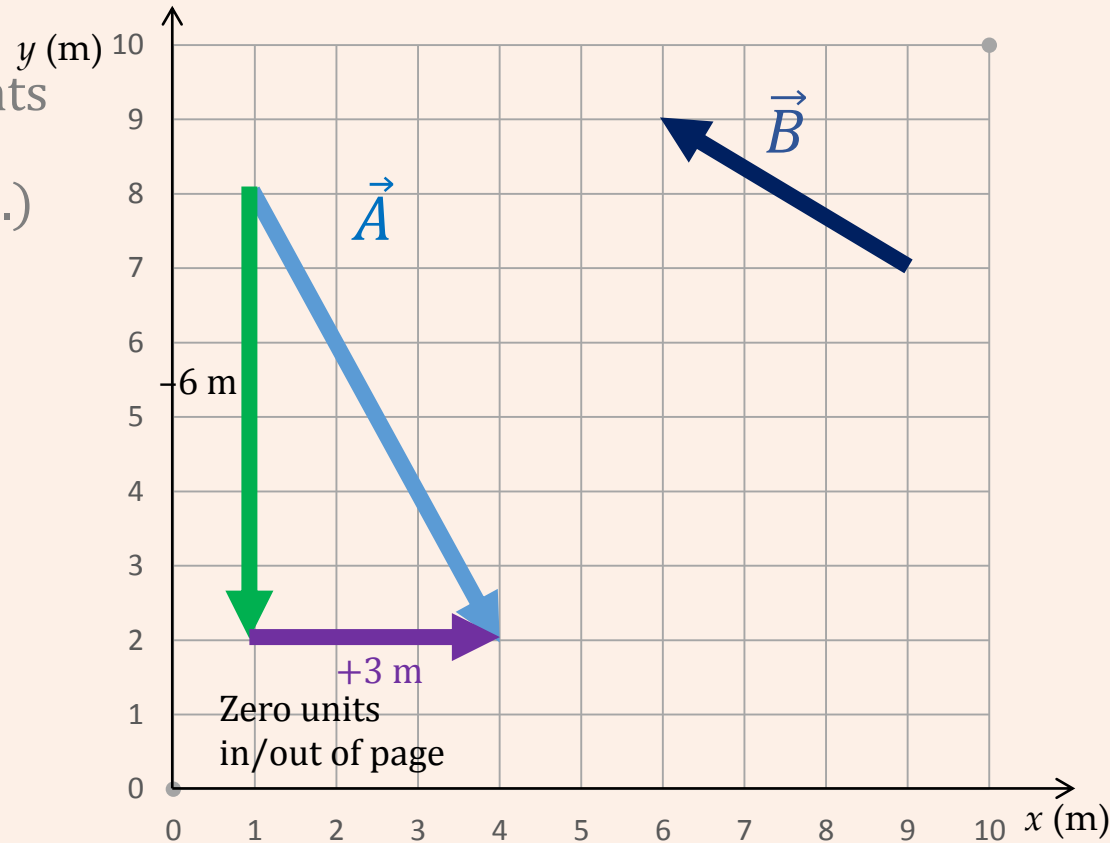
Q2. Sketch the vector:

$$\vec{B} = \langle -3, 2, 0 \rangle \text{ m.}$$

You can draw \vec{B} anywhere, as long as it goes:

- 3m in the -x direction,
- 2m in the +y direction,
- 0m in the z direction.

Q3. Calculate the magnitude of vector $\vec{B} = \langle -3, 2, 0 \rangle \text{ m}$



$$|\vec{B}| = \sqrt{B_x^2 + B_y^2 + B_z^2}$$

$$| -3, 2, 0 | = \sqrt{(-3\text{m})^2 + (2\text{m})^2 + (0\text{m})^2}$$

$$|\vec{B}| = \sqrt{13} \text{ m}$$

$$|\vec{B}| \cong 3.6 \text{ m}$$

Practice #2 Solution

Q1. Write the vector \vec{A} (shown at right), using \hat{i} , \hat{j} , \hat{k} unit vector notation:

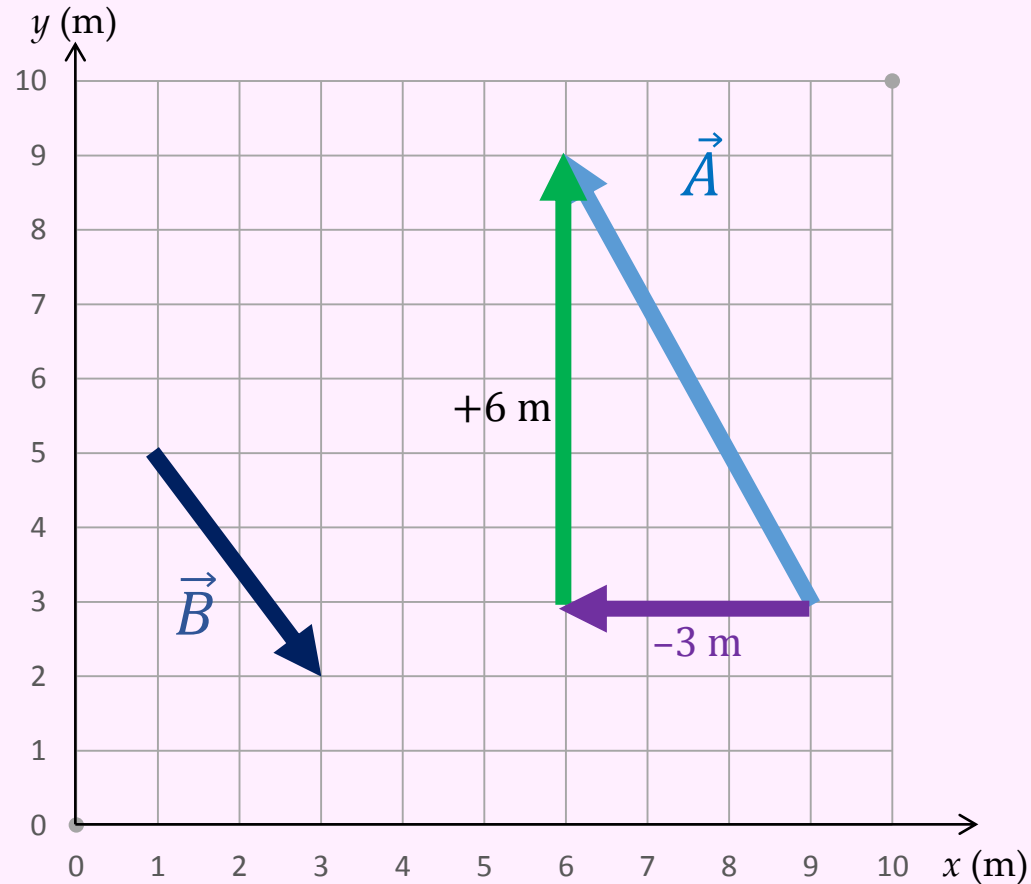
$$\vec{A} = (-3\hat{i} + 6\hat{j} + 0\hat{k}) \text{ m}$$

Q2. Given $\vec{B} = (2\hat{i} - 3\hat{j} + 0\hat{k}) \text{ m}$, sketch \vec{B} :

(Note that $B_x = 2 \text{ m}$, $B_y = -3 \text{ m}$, $B_z = 0 \text{ m}$)

You can draw $\vec{B} = (2\hat{i} - 3\hat{j} + 0\hat{k}) \text{ m}$ anywhere, as long as it goes:

- 2 meters in the $+x$ direction,
- 3 meters in the $-y$ direction,
- 0 meters in the z direction.



Practice #3 Solution

Q1. Is $\vec{c} = \left\langle -\frac{\sqrt{3}}{2}, -\frac{1}{2}, 0 \right\rangle$ a unit vector? **YES**

$$|\vec{c}| = \sqrt{c_x^2 + c_y^2 + c_z^2}$$

$$|\vec{c}| = \sqrt{\left(-\frac{\sqrt{3}}{2}\right)^2 + \left(-\frac{1}{2}\right)^2 + (0)^2}$$

$$|\vec{c}| = 1$$

The magnitude of vector \vec{c} is 1, so it **IS** a unit vector.

Q2. Is $\vec{d} = (1\hat{i} + 1\hat{j})$ a unit vector? **NO**

$$|\vec{d}| = \sqrt{d_x^2 + d_y^2 + d_z^2}$$

$$|\vec{d}| = \sqrt{(1)^2 + (1)^2 + (0)^2}$$

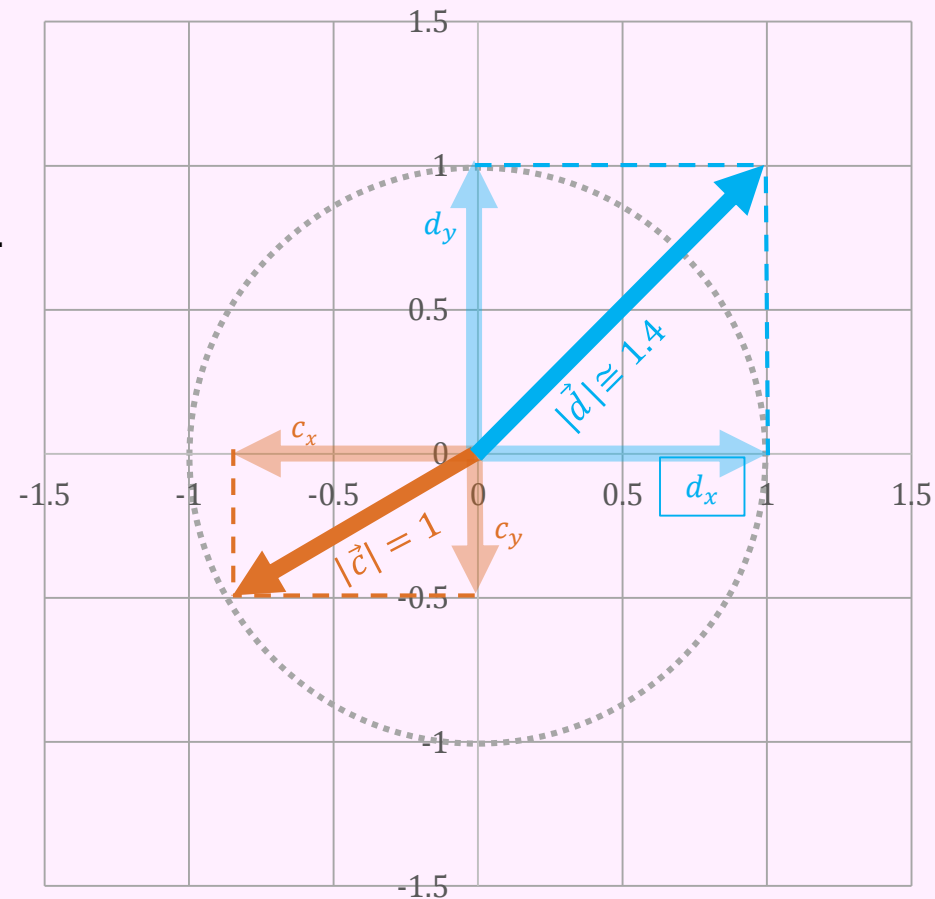
$$|\vec{d}| = \sqrt{2}$$

$$|\vec{d}| \cong 1.4$$

Because vector \vec{d} has a magnitude **larger** than 1, it is **NOT** a unit vector.

Remember, the **magnitude** of every unit vector is equal to 1

Graphical Representation



Practice #3 Solution Continued

Q3. Find the unit vector in the direction of $\vec{a} = \left\langle \frac{3}{2}, -1, 0 \right\rangle$ m.

$$\hat{a} = \frac{\vec{a}}{|\vec{a}|}$$

First, find the magnitude of \vec{a} :

$$|\vec{a}| = \sqrt{a_x^2 + a_y^2 + a_z^2}$$

$$|\vec{a}| = \sqrt{\left(\frac{3}{2}\right)^2 + (-1)^2 + (0)^2}$$

$$|\vec{a}| = \frac{\sqrt{13}}{2} \text{ m}$$

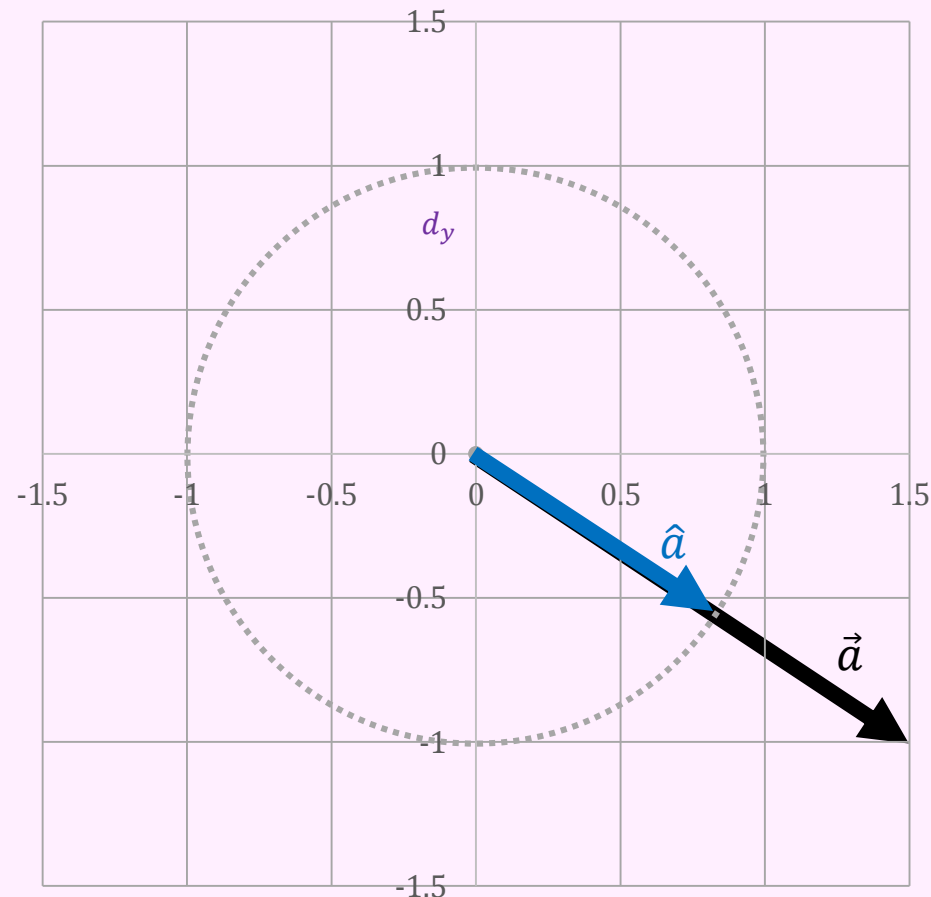
$$\hat{a} = \frac{\left\langle \frac{3}{2}, -1, 0 \right\rangle \text{ m}}{\frac{\sqrt{13}}{2} \text{ m}}$$

Note that the units cancel out. \hat{a} is dimensionless.

$$\hat{a} = \left\langle \frac{3}{\sqrt{13}}, -\frac{2}{\sqrt{13}}, 0 \right\rangle$$

$$\hat{a} \cong \langle 0.832, -0.555, 0 \rangle$$

Graphical Representation



Practice #4 Solution

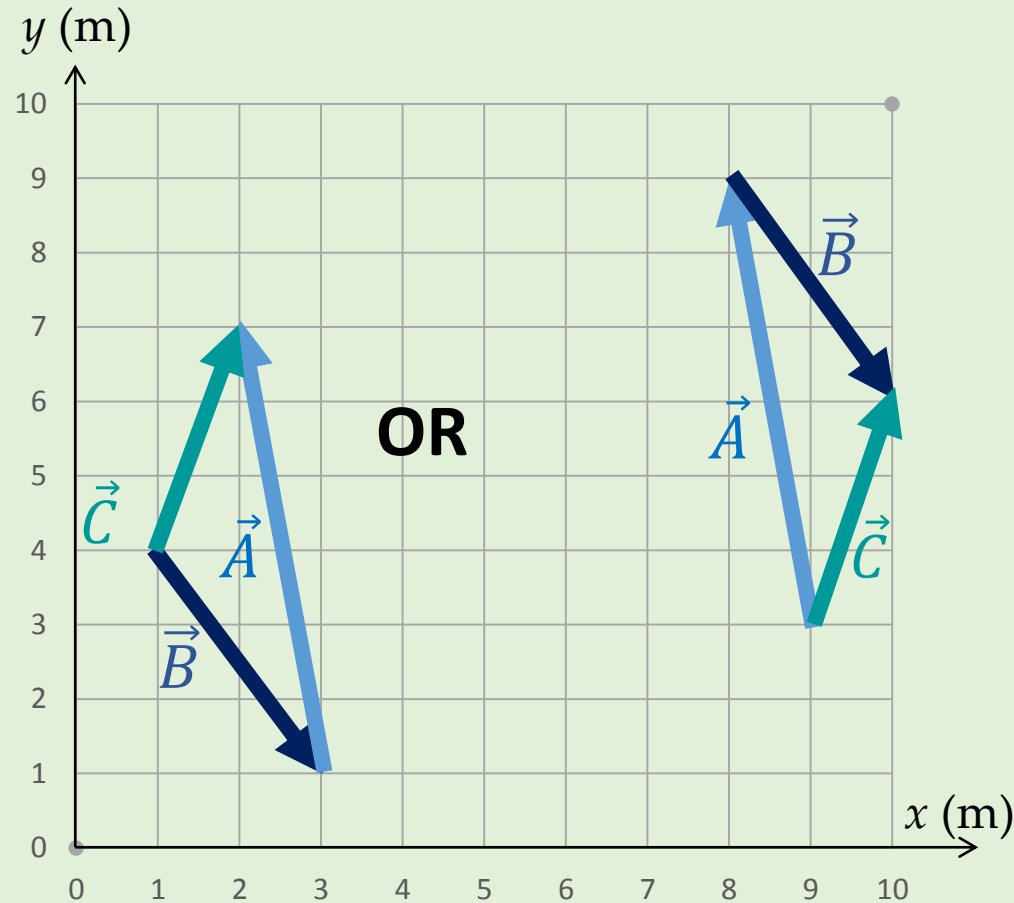
Q1. Use graphical vector addition to draw the vector $\vec{C} = \vec{A} + \vec{B}$

Step 1: Move \vec{A} or \vec{B} tip to tail

Step 2: Draw vector \vec{C} (Start from the first tail in your “tip-to-tail” chain of vectors and end at the last tip).

Vector \vec{C} should go:

1 unit in the +x direction, and
3 units in the +y direction.



Practice #5 Solution

Q1. Write vectors \vec{A} and \vec{B} in unit-vector notation

$$\vec{A} = (-1\hat{i} + 6\hat{j} + 0\hat{k}) \text{ m}$$

Because $A_x = -1 \text{ m}$, $A_y = 6 \text{ m}$, $A_z = 0 \text{ m}$

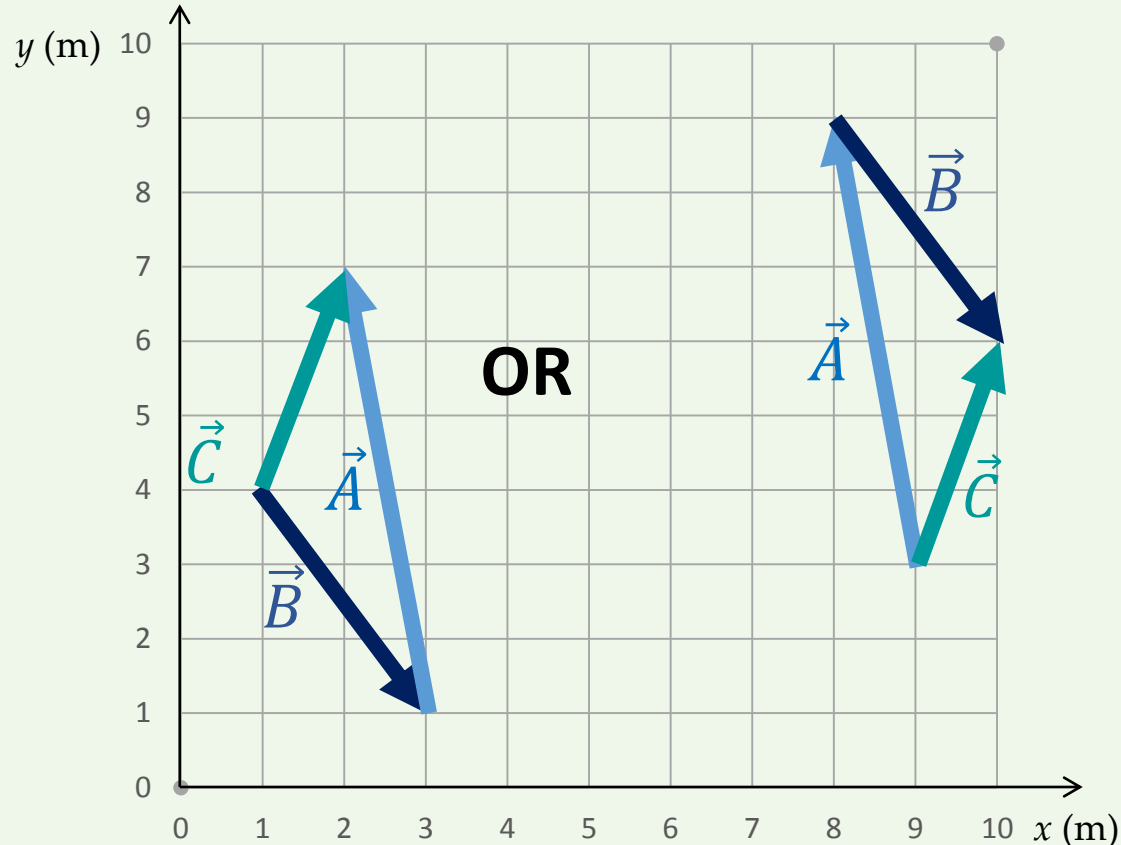
$$\vec{B} = (2\hat{i} - 3\hat{j} + 0\hat{k}) \text{ m}$$

Because $B_x = 2 \text{ m}$, $B_y = -3 \text{ m}$, $B_z = 0 \text{ m}$

Q2. Use vector addition by components to find the vector: $\vec{C} = \vec{A} + \vec{B}$

$$\begin{array}{r} \vec{A} \\ + \vec{B} \\ \hline \vec{C} \end{array} \quad \begin{array}{l} (-1\hat{i} + 6\hat{j} + 0\hat{k}) \text{ m} \\ + (2\hat{i} - 3\hat{j} + 0\hat{k}) \text{ m} \\ \hline (1\hat{i} + 3\hat{j} + 0\hat{k}) \text{ m} \end{array}$$

$$\vec{C} = (1\hat{i} + 3\hat{j} + 0\hat{k}) \text{ m}$$



Q3. Does your answer match the graphical addition of these vectors in Practice #3?

Yes, it matches! Our vector \vec{C} goes:

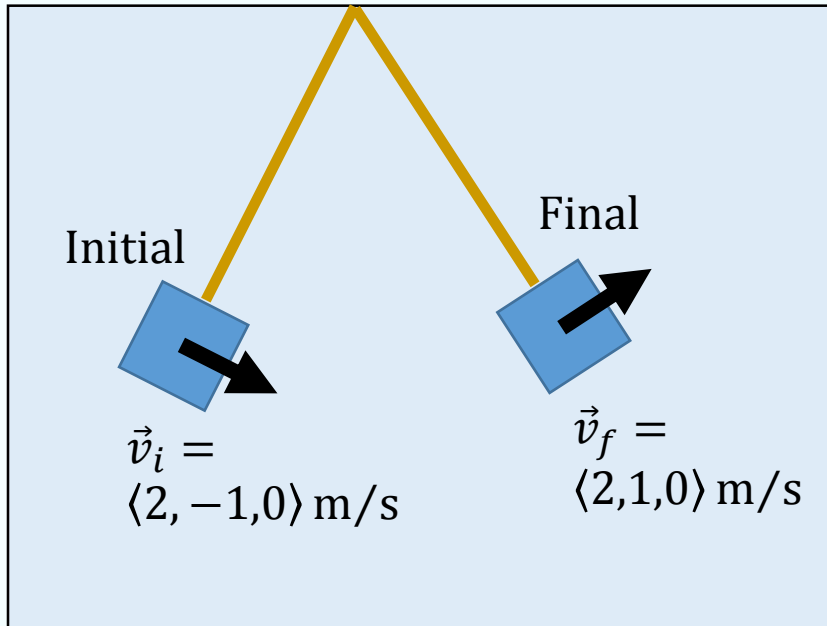
- 1 meter in the +x direction,
- 3 meters in the +y direction, and
- 0 meters out of the page.

Practice #6 Solution

Q1. The initial and final velocity vectors are given in the figure.

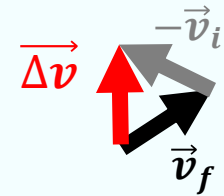
We define $\overrightarrow{\Delta v} = \overrightarrow{v_f} - \overrightarrow{v_i}$. For this scenario, find $\overrightarrow{\Delta v}$:

- graphically and
- by components.



Graphically:

$$\overrightarrow{\Delta v} = \overrightarrow{v_f} + (-\overrightarrow{v_i})$$



By components:

$$\overrightarrow{\Delta v} = \overrightarrow{v_f} - \overrightarrow{v_i}$$

$$\overrightarrow{v_f} \quad \langle 2, 1, 0 \rangle \text{ m/s}$$

$$-\overrightarrow{v_i} \quad - \langle 2, -1, 0 \rangle \text{ m/s}$$

$$\overrightarrow{\Delta v} \quad \langle 0, 2, 0 \rangle \text{ m/s}$$