Vectors II

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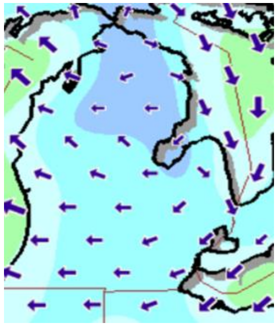
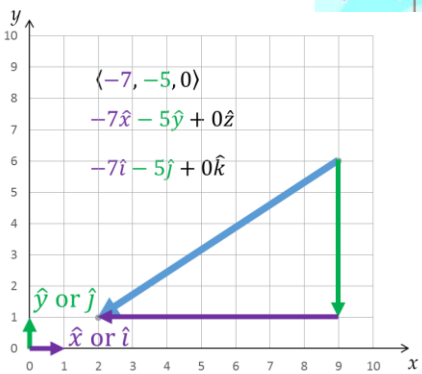
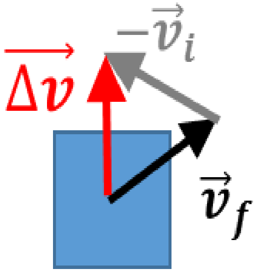
Recap of Vectors I

After working through the Vectors I module, we expect that you can already:

- Distinguish vector quantities from scalar quantities
- Represent vectors graphically and mathematically (including using unit-vector notation)
- Use vector components to find the magnitude of a vector
- Add and subtract vectors graphically and by using components
- Multiply vectors by scalars



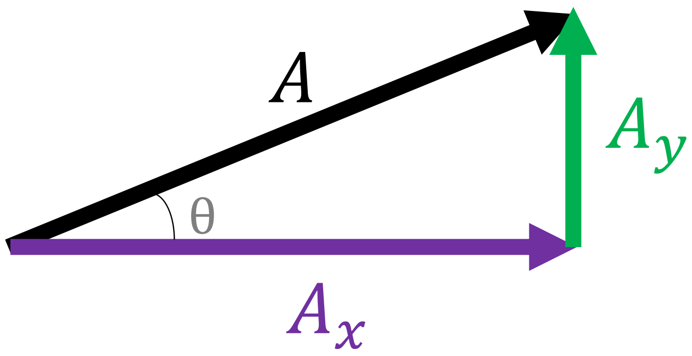




Goals for Vectors II

After working through this module, you will be able to:

- Calculate a vector's components from its magnitude and direction
- Calculate a vector's magnitude and direction from its components
- Correctly add vectors when given their magnitudes and directions
- Find components of vectors in tilted coordinate systems



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Calculating a Vector's Components

From its Magnitude and Direction

Calculating Components of a Vector

If you know a vector's magnitude and direction in a coordinate system, you can use trigonometry to find its x- and y- components.

$$\cos(\theta) = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$\cos(\theta) = \frac{A_x}{A}$$

$$A_x = A\cos(\theta)$$

$$\sin(\theta) = \frac{\text{opposite}}{\text{hypotenuse}}$$

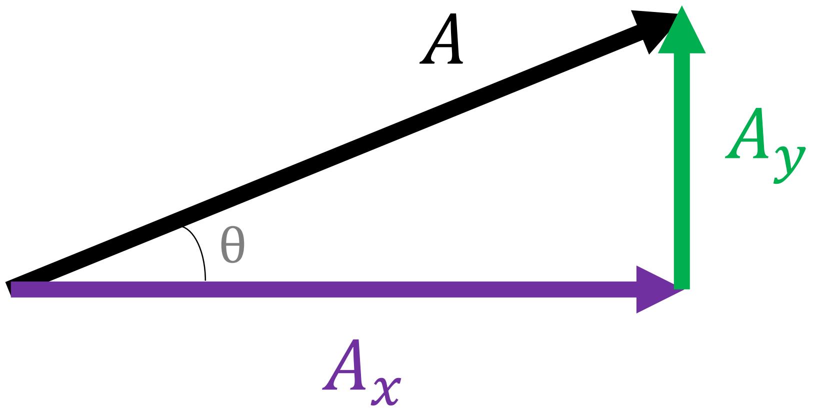
$$\sin(\theta) = \frac{A_y}{A}$$

$$A_y = A\sin(\theta)$$

Remember!

- "adjacent" is the side next to the angle
- "opposite" is the side across from the angle

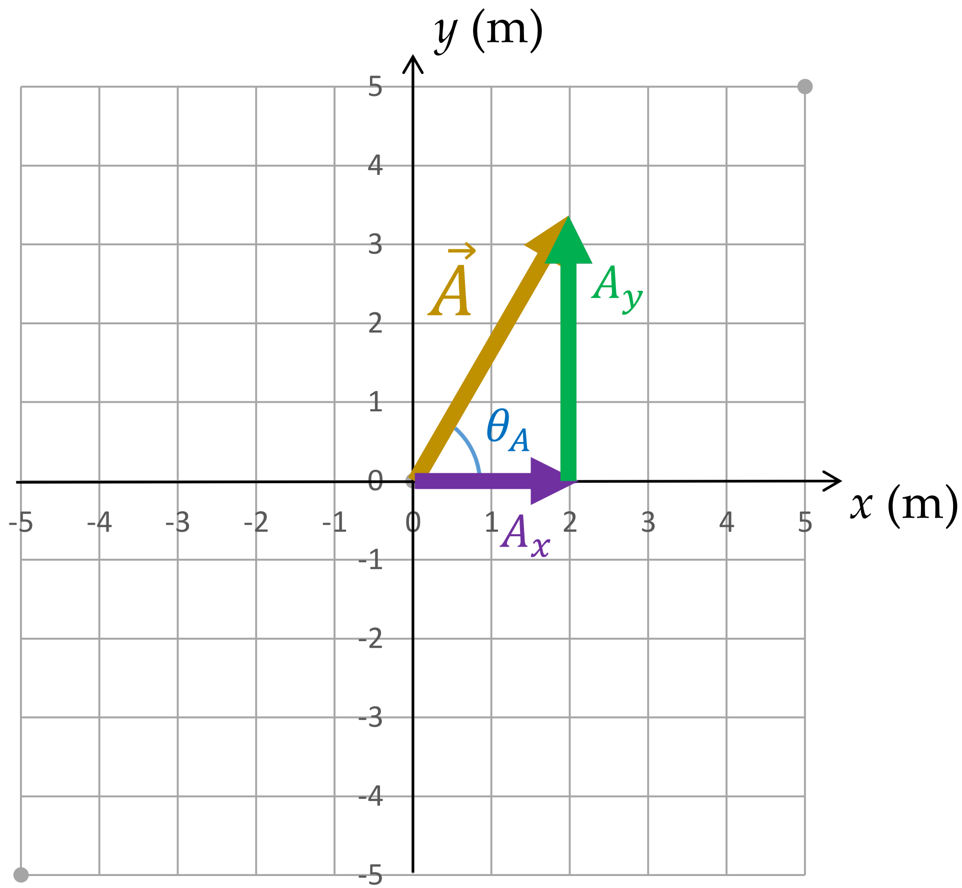
Note: θ is measured from the $+x$ axis



Calculating Components

Worked Example:

Vector $\vec{A}$ has a magnitude of 4 meters ($A = 4$ m) and is oriented at an angle $\theta_A = 60°$ counter-clockwise from the $+x$ axis. Find A_x and A_y.



See next page for solution

Calculating Components

Worked Example:

Vector $\vec{A}$ has a magnitude of 4 meters ($A = 4$ m) and is oriented at an angle $\theta_A = 60°$ counter-clockwise from the $+x$ axis. Find A_x and A_y.

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$\cos\theta_A = \frac{A_x}{A}$$

$$A\cos\theta_A = A_x$$

$$(4\text{ m})\cos 60° = A_x$$

$$\boxed{2\text{ m} = A_x}$$

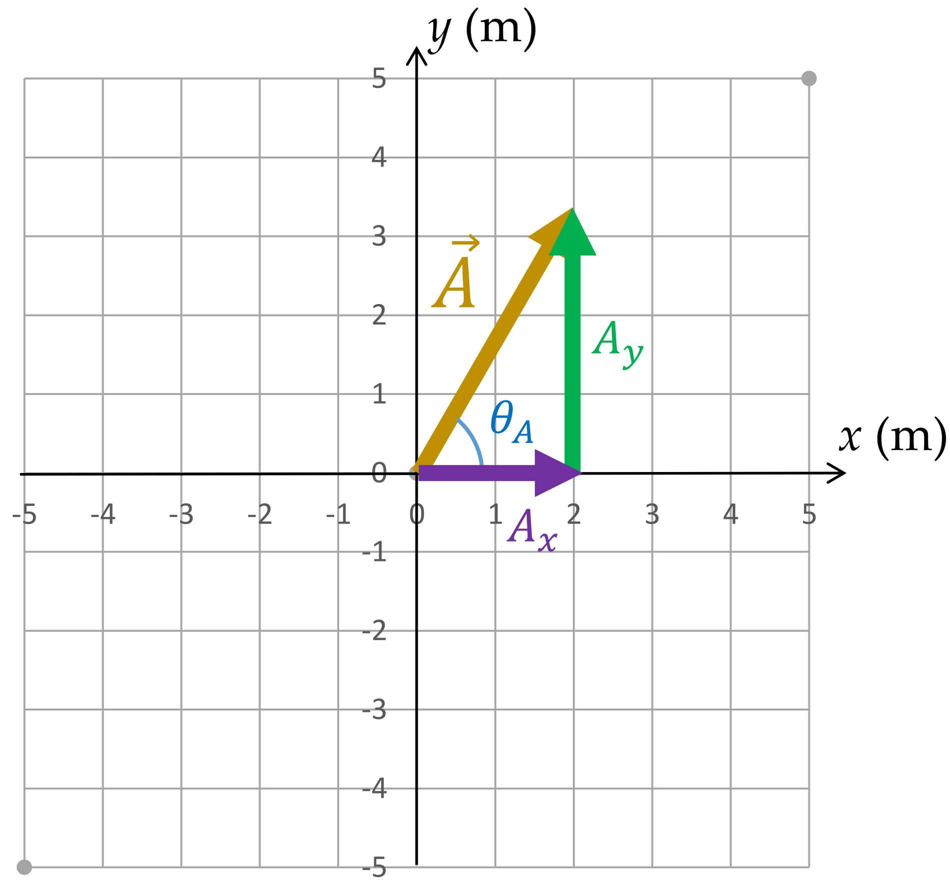
$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$\sin\theta_A = \frac{A_y}{A}$$

$$A\sin\theta_A = A_y$$

$$(4\text{ m})\sin 60° = A_y$$

$$\boxed{2\sqrt{3}\text{ m} = A_y}$$



$$\vec{A} = \langle 2, 2\sqrt{3}\rangle\text{ m}$$

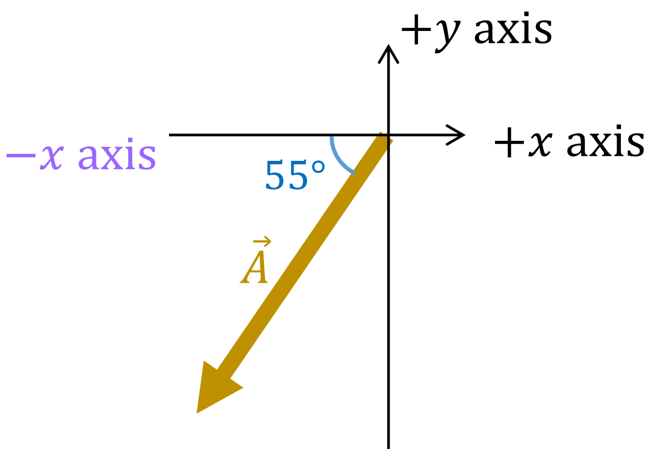
or

$$\vec{A} = (2\hat{\imath} + 2\sqrt{3}\hat{\jmath})\text{ m}$$

Vector Direction

What if the given angle is not measured from the $+x$ axis?

Let's say we are given that $\vec{A}$ has a magnitude of 6 meters and direction 55° counterclockwise (CCW) from the $-x$ axis.



What angle does $\vec{A}$ make with the $+x$ axis?

Vector Direction

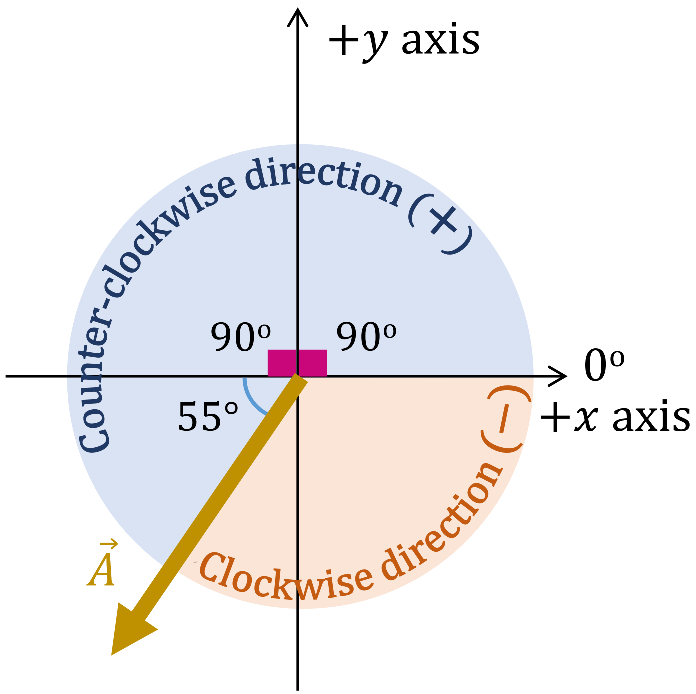
What angle does $\vec{A}$ make with the $+x$ axis?

$$90° + 90° + 55° = 235°$$

$\vec{A}$ has a direction of $+235°$ counterclockwise from the $+x$ axis.

We could also report the direction of $\vec{A}$ as $-125°$ (clockwise from the $+x$ axis)

$$235° - 360° = -125°$$



Calculating Components

We can use either of the angles measured from the $+x$ axis to calculate the components of $\vec{A}$:

$$A\cos(235°) = A_x$$

OR

$$A\cos(-125°) = A_x$$

$$(6\text{ m})\,(-0.574) = A_x$$

$$\boxed{-3.4\text{ m} \cong A_x}$$

$$A\sin(235°) = A_y$$

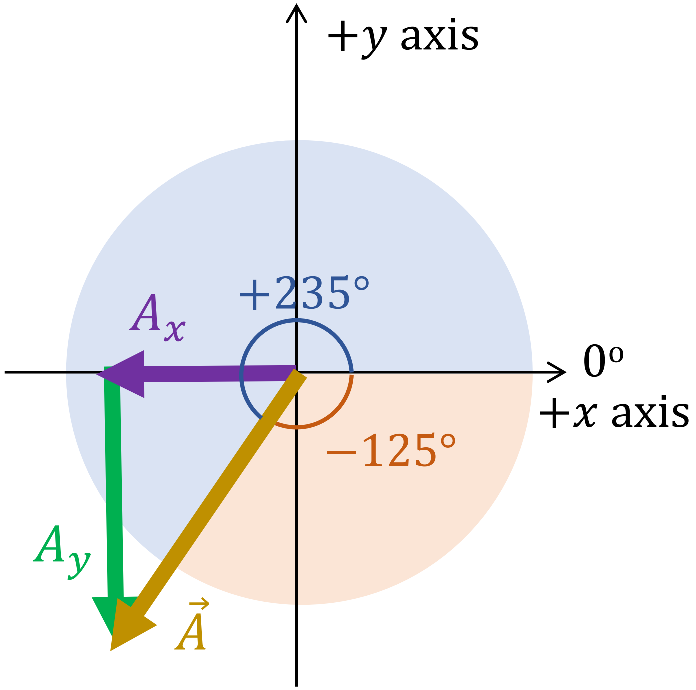
OR

$$A\sin(-125°) = A_y$$

$$(6\text{ m})\,(-0.819) = A_y$$

$$\boxed{-4.9\text{ m} \cong A_y}$$

Note: When using the angle from the $+x$ axis, the calculator gives the correct positive and negative signs.



Calculating Components - Shortcut

You can use a "shortcut" angle between the vector and a nearby axis, but **you** must assign the appropriate positive and negative signs.

In this example, using the original 55° angle CCW from the $-x$ axis:

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$\cos\theta = \frac{A_x}{A}$$

$$A\cos\theta = A_x$$

$$(6\text{ m})\cos 55° = A_x$$

$$3.4\text{ m} \cong A_x$$

A_x points in the negative x direction, so we give it a negative sign:

$$A_x \cong -3.4\text{ m}$$

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$\sin\theta = \frac{A_y}{A}$$

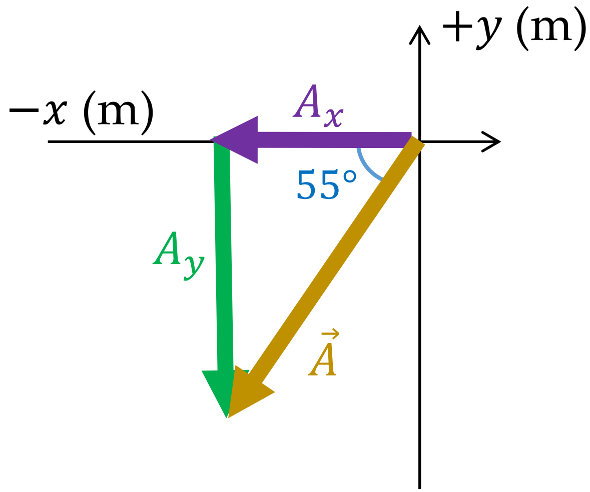
$$A\sin\theta = A_y$$

$$(6\text{ m})\sin 55° = A_y$$

$$4.9\text{ m} \cong A_y$$

A_y points in the negative y direction, so we give it a negative sign:

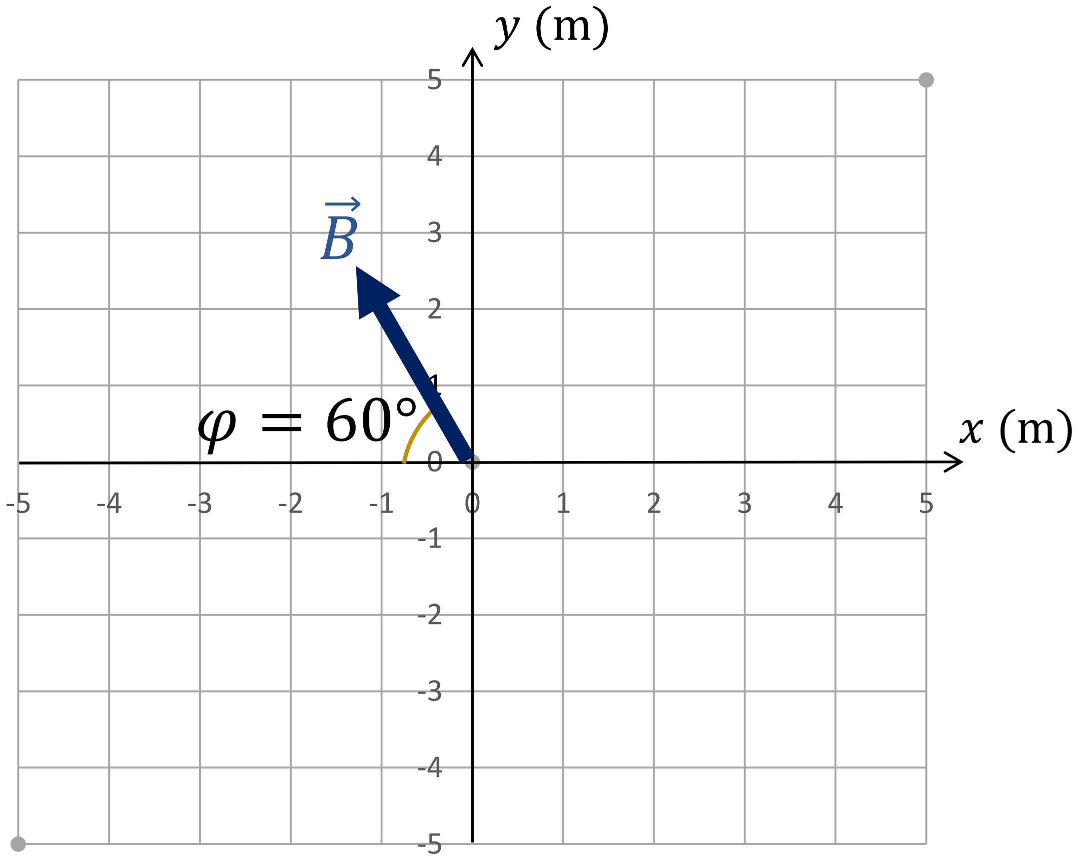
$$A_y \cong -4.9\text{ m}$$



Note that this matches the previous method!

Practice #1

Vector $\vec{B}$ has a magnitude of 3 meters ($B = 3$ m) and is oriented at an angle 60° clockwise from the $-x$ axis. Find B_x and B_y.



See end of packet for solutions

Calculating a Vector's Magnitude and Direction

From its Components

Finding the Magnitude from Components

Remember from the *Vectors I* module, we can find a vector's magnitude if we know its components.

If we are given:

$A_x = -2\text{ m}$

$A_y = -2\sqrt{3}\text{ m}$

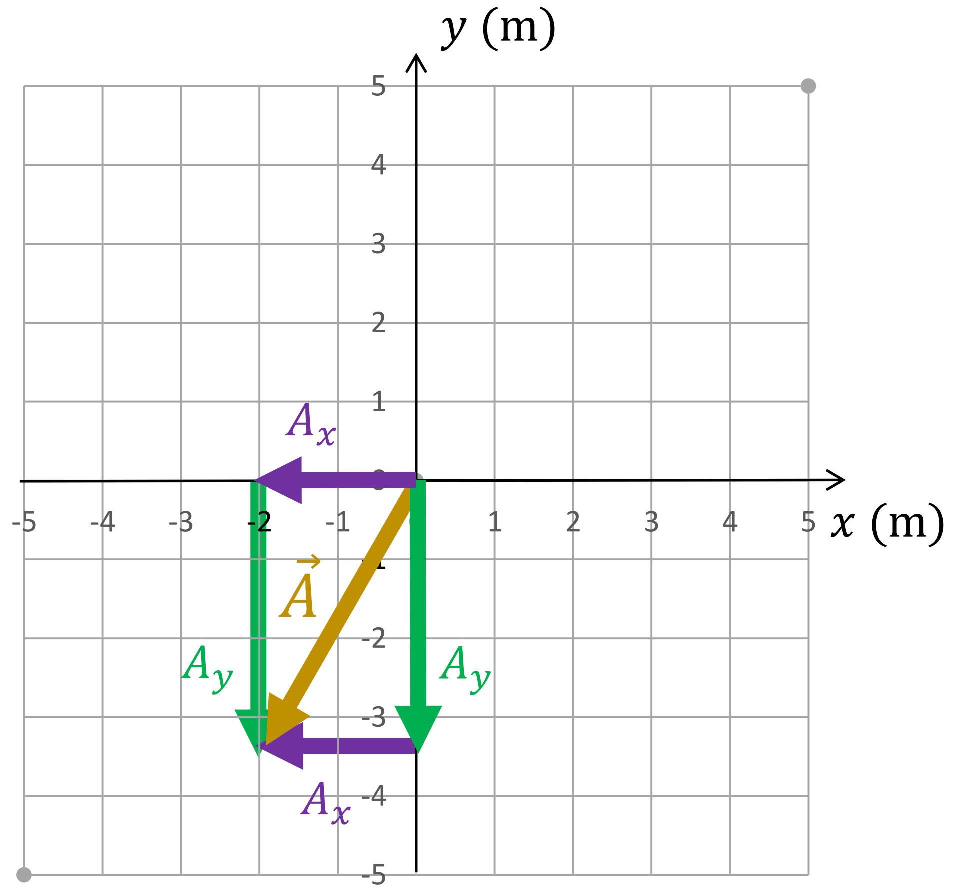
$A_z = 0\text{ m}$

We use the Pythagorean Theorem to calculate the magnitude of $\vec{A}$:

$$A = \sqrt{A_x^2 + A_y^2 + A_z^2}$$

$$A = \sqrt{(-2\text{ m})^2 + \left(-2\sqrt{3}\text{ m}\right)^2 + (0\text{ m})^2}$$

$$\boxed{A = 4\text{ m}}$$



Finding the Angle from Components

If we know the components, we find the direction using trigonometry.

Given $A_x = -2\text{ m}$, $A_y = -2\sqrt{3}\text{ m}$, and $A_z = 0\text{ m}$:

$$\tan\varphi = \frac{\text{opposite}}{\text{adjacent}}$$

$$\tan\varphi = \left|\frac{A_y}{A_x}\right|$$

$$\tan\varphi = \left|\frac{-2\sqrt{3}\text{ m}}{-2\text{ m}}\right|$$

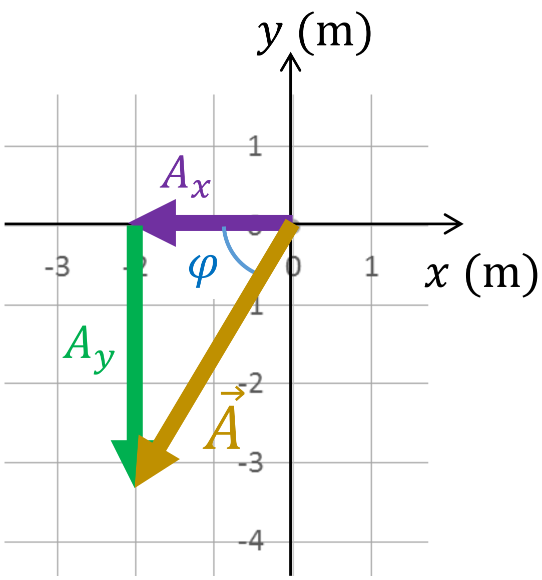
$$\tan\varphi = \frac{2\sqrt{3}}{2}$$

$$\tan\varphi = \sqrt{3}$$

$$\varphi = \tan^{-1}\sqrt{3}$$

$$\varphi = 60°$$

(CCW from –x axis)



The angle from the $+x$ axis would be:

$180° + 60° = \boxed{240°}$

⚠ Caution: Your calculator doesn't always give the angle from the $+x$ axis.

Make sure you **sketch** the vector, and think carefully about which angle you are finding.

Your calculator doesn't know the difference between $\tan^{-1}\left(\frac{-2\sqrt{3}}{-2}\right)$ and $\tan^{-1}\left(\frac{2\sqrt{3}}{2}\right)$. (Try it!)

“Magnitude-Angle” Notation

For the previous example, we have found that the magnitude is $A = 4$ m and that the direction is $\theta = 240°$. An alternate way to write this out is:

$$\text{magnitude} \angle \text{direction}.$$

For our example this looks like:

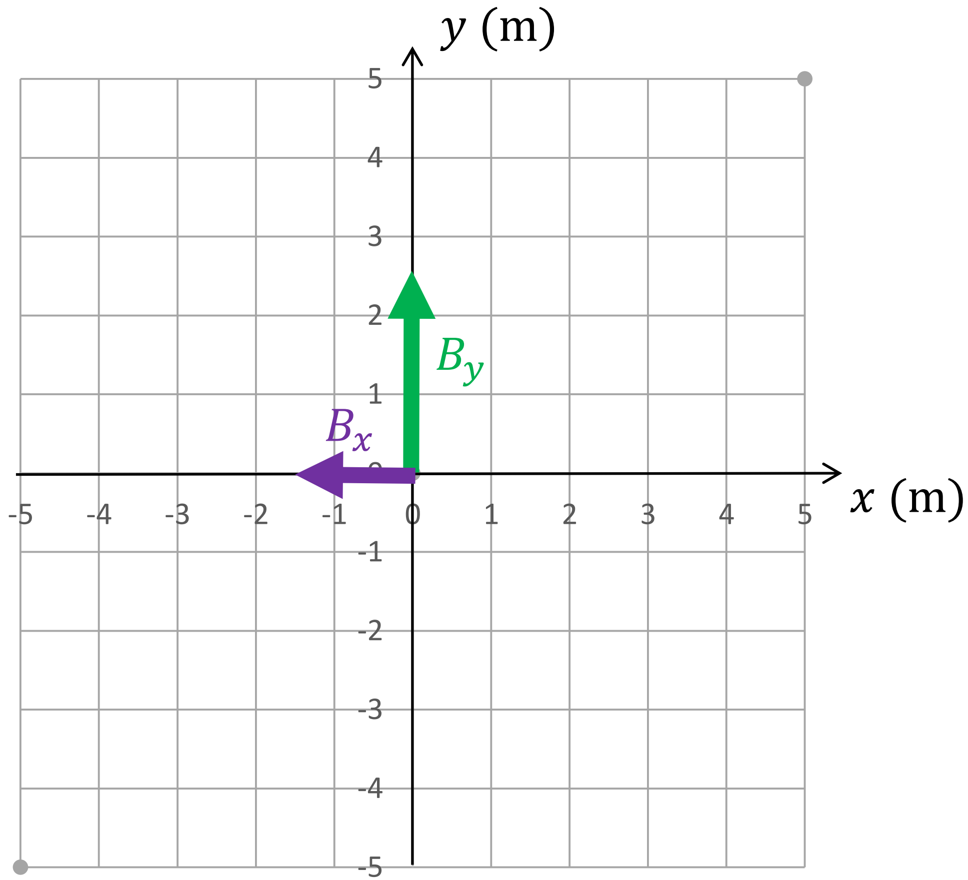
$$\vec{A} = A \angle \theta$$

$$\vec{A} = 4 \text{ m} \angle 240°$$

This is known as “polar” or “magnitude-angle” notation.

Practice #2

Find the magnitude and direction of vector $\vec{B} = \left(-\frac{3}{2}\,\hat{\imath} + 3\frac{\sqrt{3}}{2}\hat{\jmath}\right)$ m.



See end of packet for solutions

Comprehensive Practice #3

This problem requires concepts from both Vectors I and Vectors II modules.

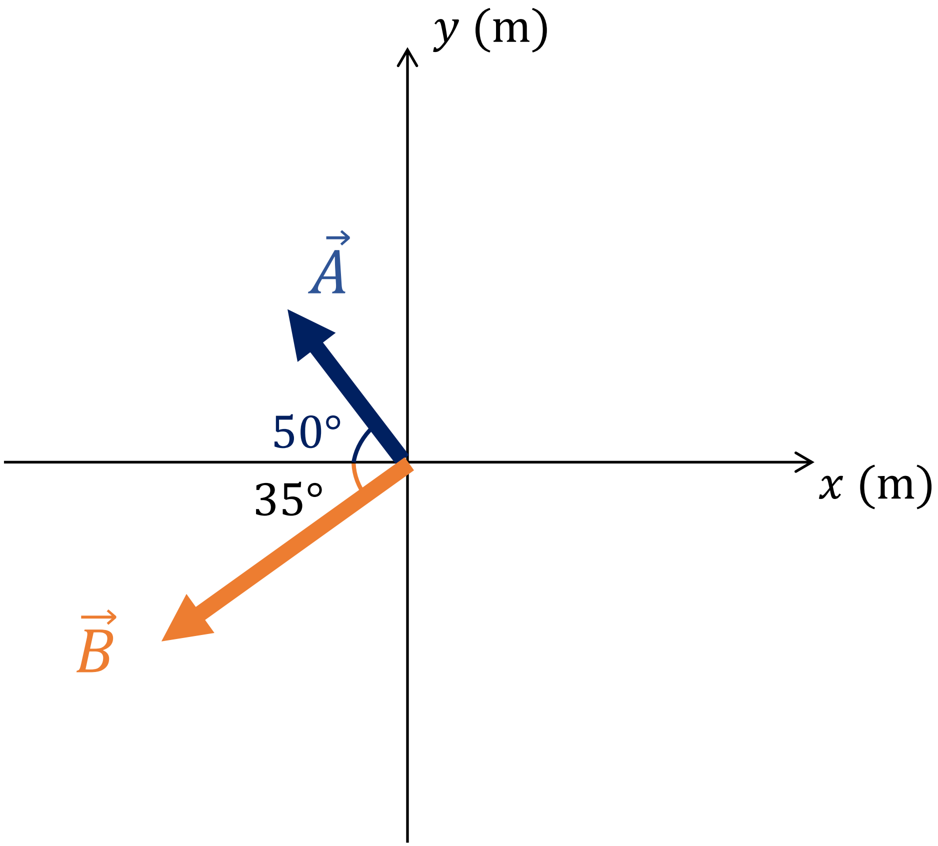
Given:

A = 2.5 m
50° clockwise from $-x$ axis

B = 4 m
35° counterclockwise from $-x$ axis

Find the magnitude and direction of the vector $\vec{C} = \vec{A} + \vec{B}$

- First, find the components of $\vec{A}$ and $\vec{B}$
- Next, do the addition
- Then calculate the magnitude and direction of the resulting vector $\vec{C}$



See end of packet for solutions

Calculating a Vector's Components

in a Tilted Coordinate System

Tilted Axes: Why?!

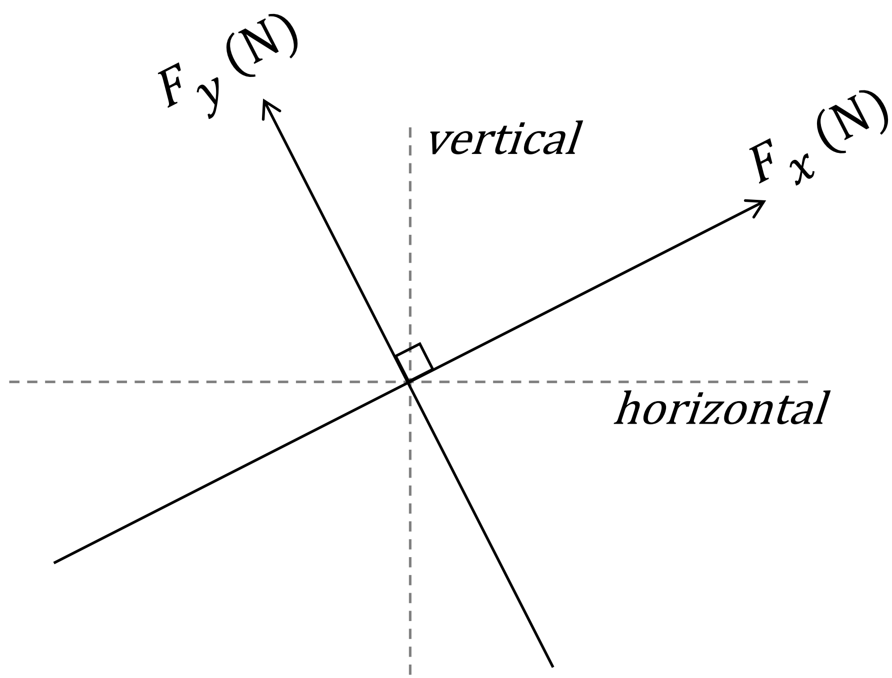
You can put your axes anywhere, as long as the axes stay perpendicular to each other.

Often, it's not convenient to use horizontal and vertical axes.

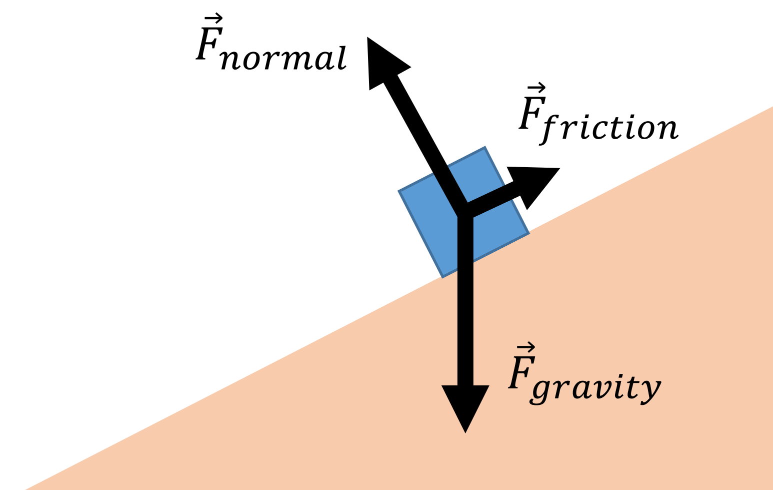
- It is easier to have one axis in the direction of the acceleration vector. On ramps/hills, that is usually parallel to the surface.

"**Vertical**" means straight toward or away from the center of the earth.

"**Horizontal**" means parallel to the horizon.



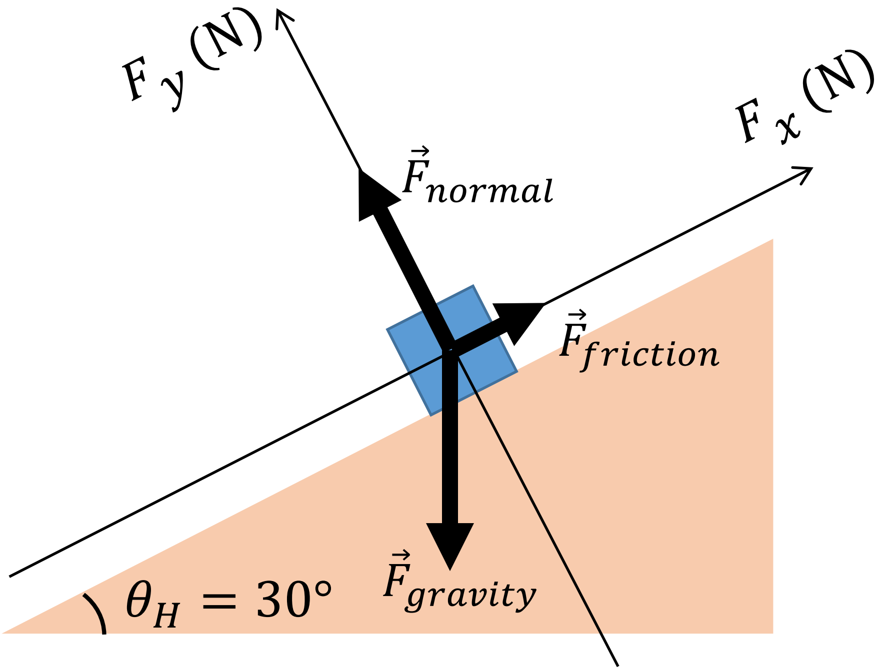
Force vectors



Vector Components with Tilted Axes

- Notice that $\vec{F}_{normal}$ and $\vec{F}_{friction}$ are oriented exactly along the axes, so we don't need to do any calculations to find their x- or y- components.
- However, since the $\vec{F}_{gravity}$ is vertical, NOT in line with the coordinate system, we must use trigonometry to find its x- and y- components.

Let's say that a ramp is inclined at a 30° angle from the horizontal. We'll need to figure out how this angle is related to $\vec{F}_{gravity}$.

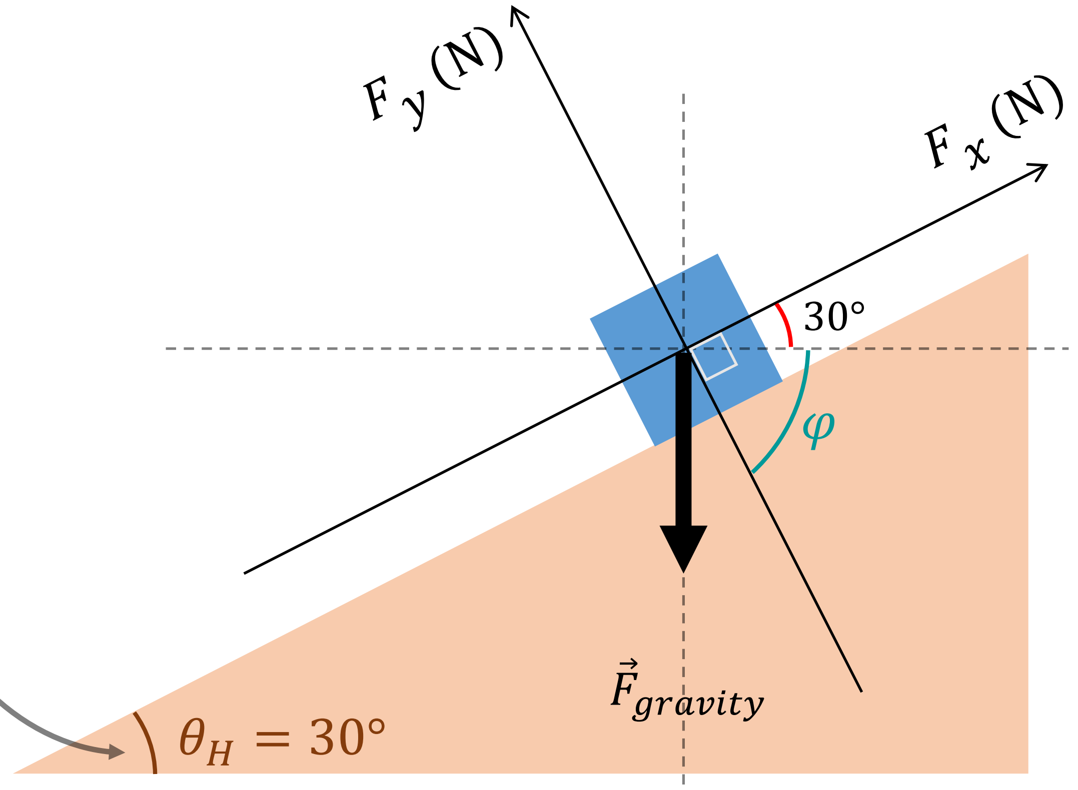


Tilted Axes: Finding Angles

- Start by describing <u>in words</u> the angle that you are given:

  "The angle between the horizontal and the surface of the ramp is 30°."

- The angle in red must also be 30°, since it is <u>**also**</u> the angle between the horizontal and the surface of the ramp.



- Since we know that the x- and y- axes are perpendicular to each other, we can figure out what the angle marked φ must be…

Tilted Axes: Finding Angles

Looking at the figure,

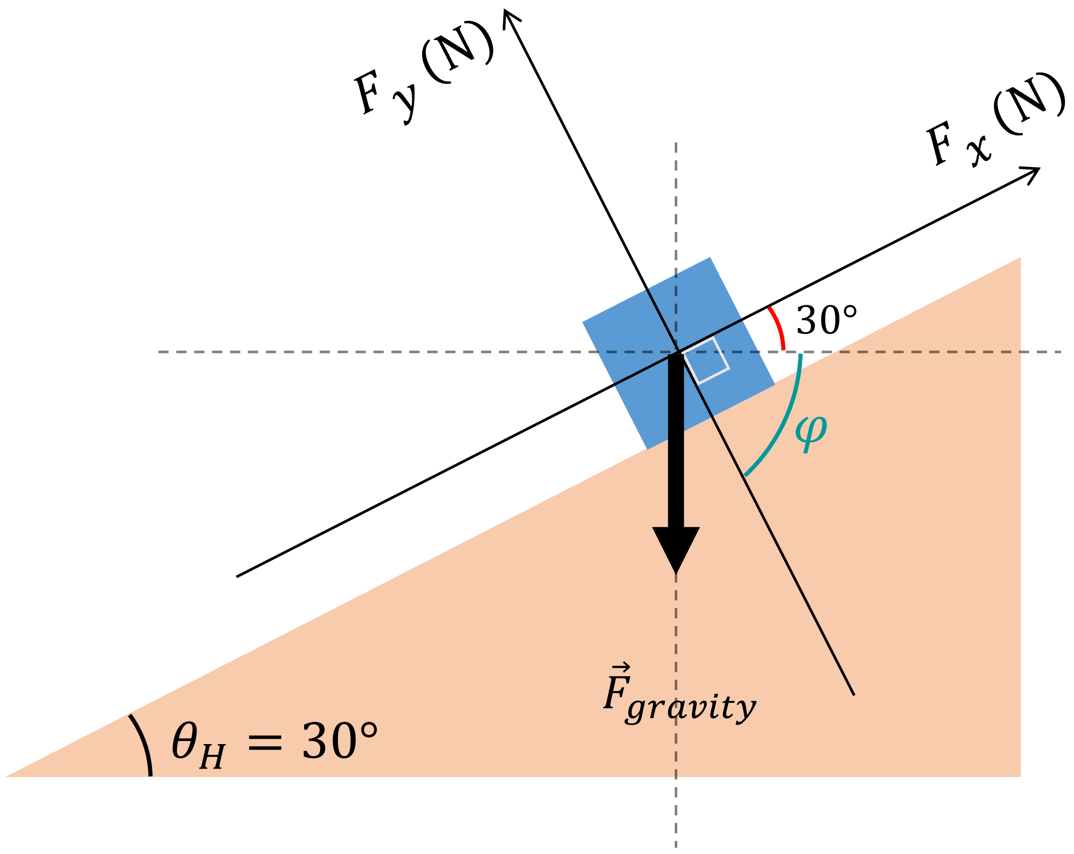
$$\varphi + 30° = 90°$$

So, solving for φ,

$$\varphi = 90° - 30°$$

$$\varphi = 60°$$

Note: There are a lot of ways to approach this. (Ask an SLC assistant for help if this way is confusing for you!)



We can use a similar method to find the angle that $\vec{F}_{gravity}$ makes with the y-axis.

Tilted Axes: Finding Angles

- We know $\vec{F}_{gravity}$ always points straight down toward the center of the earth (vertically downward).

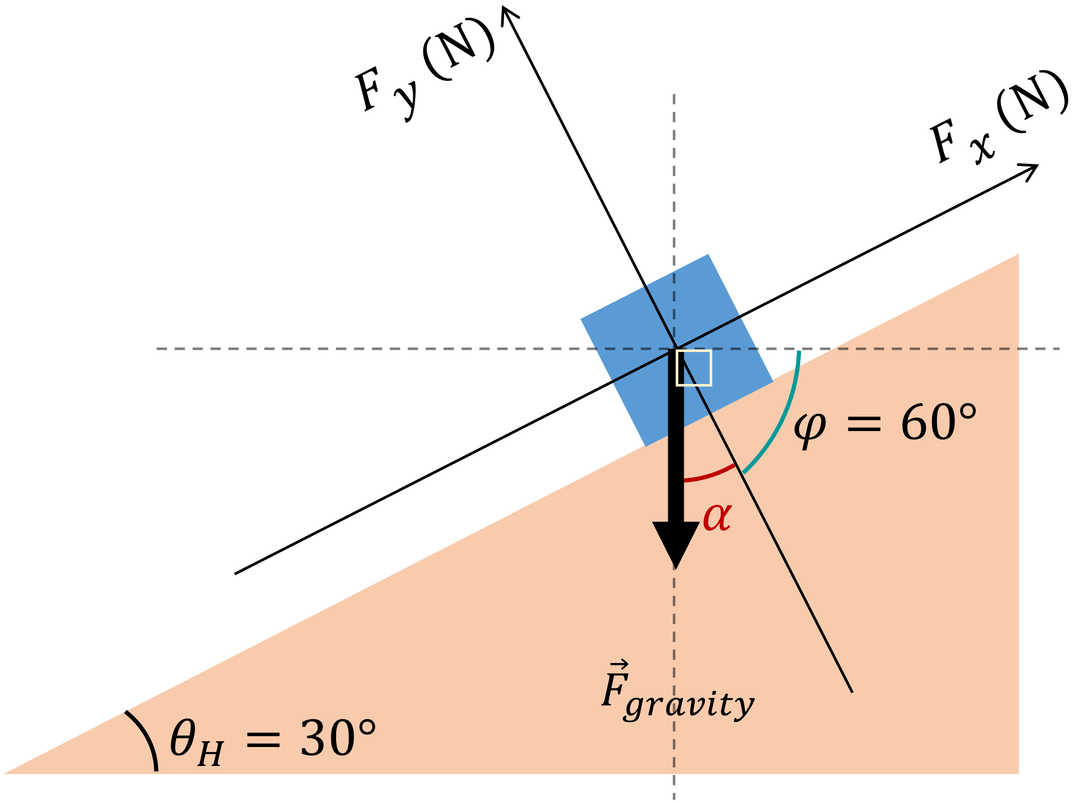
- Looking at the figure,

$$\varphi + \alpha = 90°$$

$$60° + \alpha = 90°$$

So, solving for α,

$$\alpha = 90° - 60° = 30°$$



- Now that we've found an angle between $\vec{F}_{gravity}$ and one of our axes, we can find the x- and y- components of this force.

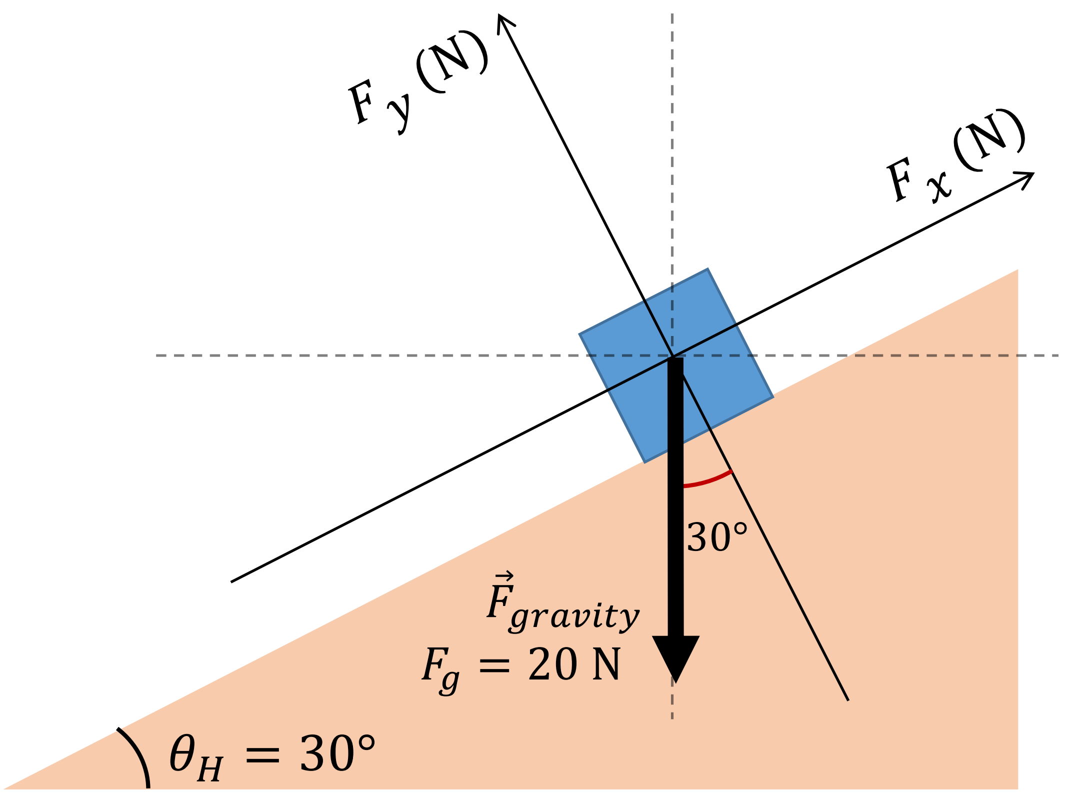
Tilted Axes: Components

Worked Example:

Let's say the magnitude of the gravitational force is 20 Newtons ($F_g = 20$ N). How would you calculate F_{g_x} and F_{g_y}?

- First, how would you sketch F_{g_x} and F_{g_y}?

See next page for solution



Tilted Axes: Components

<u>Worked Example:</u>

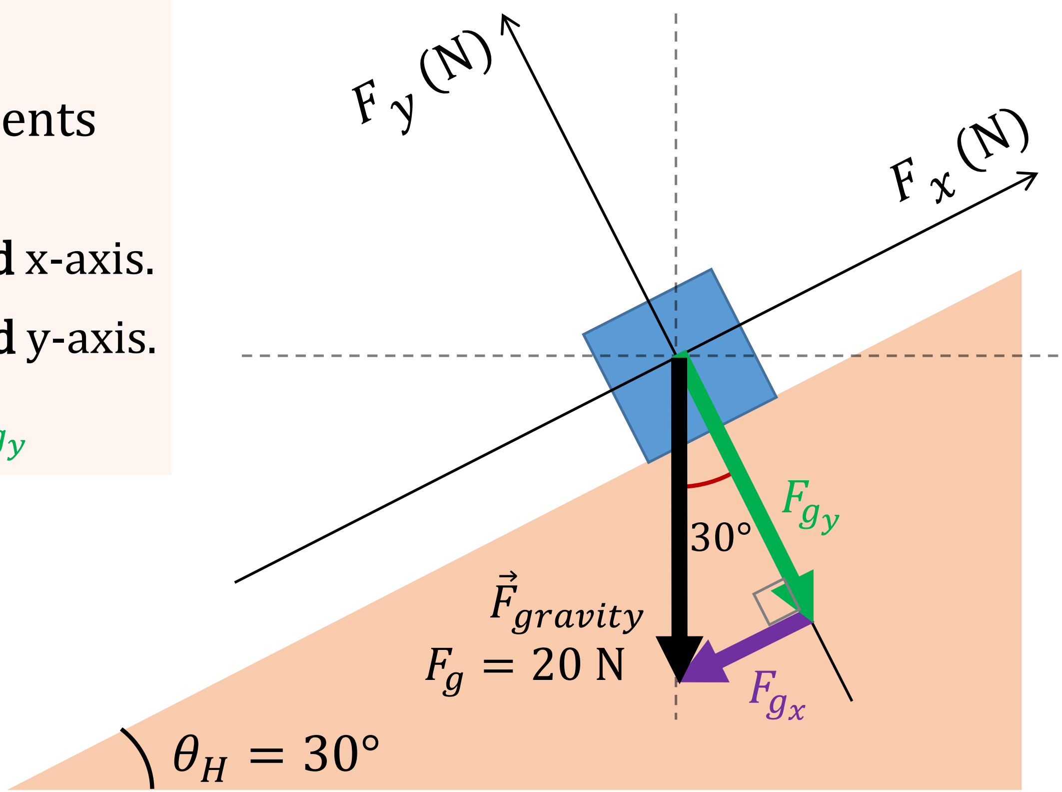
Let's say the magnitude of the gravitational force is 20 Newtons ($F_g = 20$ N). How would you calculate F_{g_x} and F_{g_y}?

- First, how would you sketch F_{g_x} and F_{g_y}?

Be careful!

Drawing the x- and y- components can be tricky:

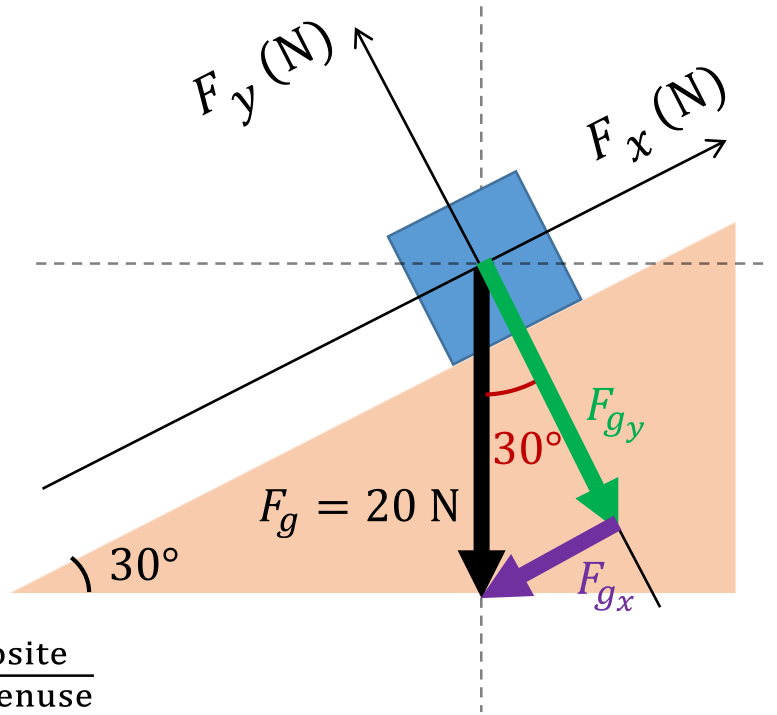
- F_{g_x} must be parallel to the **tilted** x-axis.
- F_{g_y} must be parallel to the **tilted** y-axis.
- F_{g_x} must be perpendicular to F_{g_y}



Solution continues on next page

Tilted Axes: Components

- It is common to use the 30° angle as a shortcut. (Instead of using $\theta = -120°$ from the $+x$ axis.)
- If we use the 30° shortcut, we have to put in the correct negative signs, **and** do the trig carefully:



$$\cos\alpha = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$\cos\alpha = \frac{F_{g_y}}{F_g}$$

$$F_g\cos\alpha = F_{g_y}$$

$$(20\text{ N})\cos 30° = F_{g_y}$$

$$10\sqrt{3}\text{ N} = F_{g_y}$$

$$\boxed{-10\sqrt{3}\text{ N} = F_{g_y}}$$

$$\sin\alpha = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$\sin\alpha = \frac{F_{g_x}}{F_g}$$

$$F_g\sin\alpha = F_{g_x}$$

$$(20\text{ N})\sin 30° = F_{g_x}$$

$$10\text{ N} = F_{g_x}$$

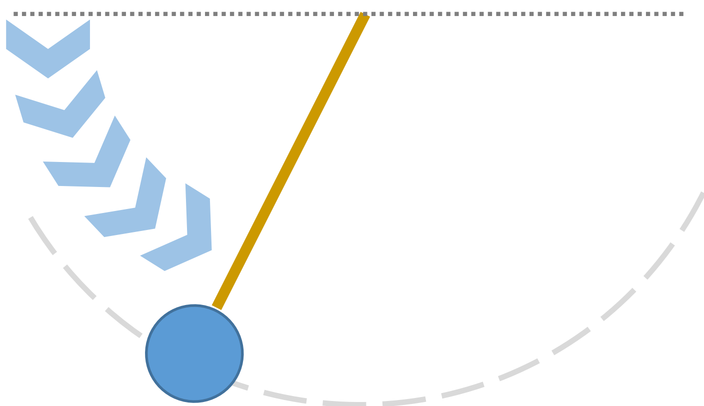
$$\boxed{-10\text{ N} = F_{g_x}}$$

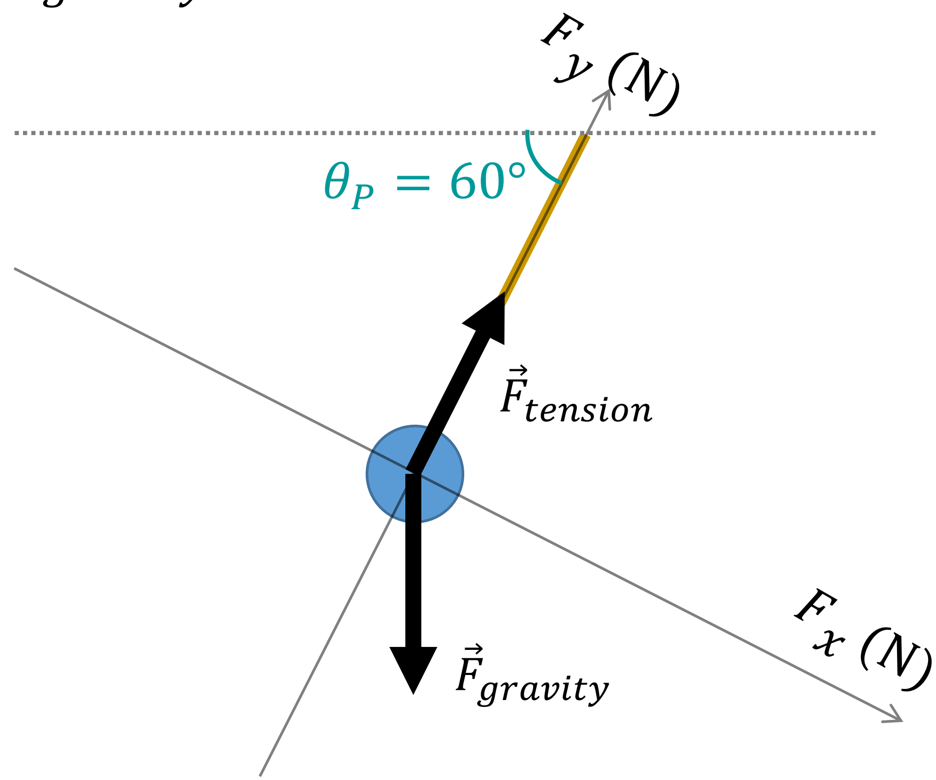
Note that when we use this 30° angle, **cos** gives F_{g_y} and **sin** gives F_{g_x}

F_{g_x} and F_{g_y} both point in the **negative direction**, so we give them **negative signs**.

Practice #4

- At a particular instant, a pendulum has swung down to an angle of 60° from the horizontal, as shown below.
- The magnitude of $\vec{F}_{gravity}$ is given as 6 Newtons.
- Find the x- and y- components of $\vec{F}_{gravity}$.

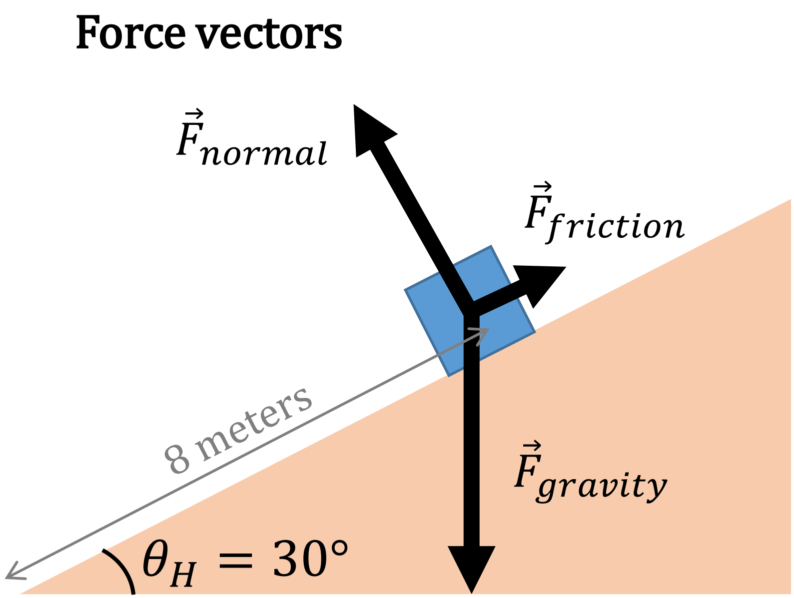




See end of packet for solutions

Relevant and Irrelevant Information

- Often in a physics problem, you're given lots of information, but you have to figure out what is relevant.
- Pay attention to units!
- Can you use the 8 meters shown in the figure to find the force of gravity?



Relevant and Irrelevant Information

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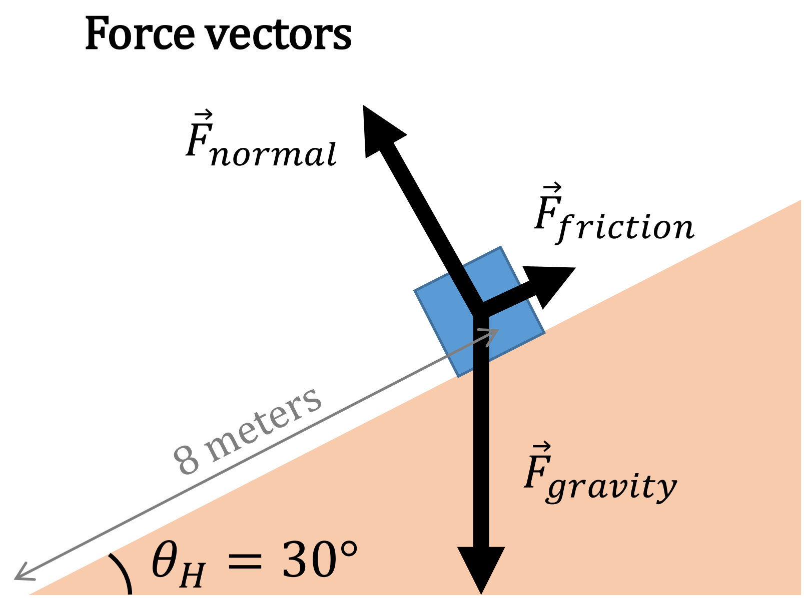


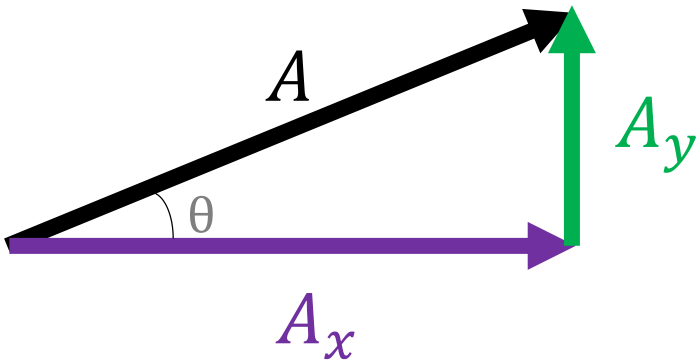
NO!

- Even though the picture is drawn so it looks like 8 meters and $\vec{F}_{gravity}$ form two sides of a triangle, they are completely different things with completely different units!
- When using trigonometry, all sides of your triangle must have the same units!!

Recap for Vectors II

After working through this module, you should now be able to:

- Calculate a vector's components from its magnitude and direction
- Calculate a vector's magnitude and direction from its components
- Correctly add vectors when given their magnitudes and directions
- Find components of vectors in tilted coordinate systems

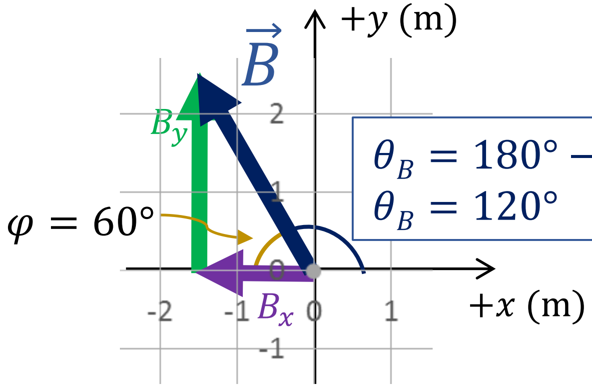


https://commons.wikimedia.org/wiki/Alpine_skiing

If you have any questions, please refer to the SLC assistants for help!

Practice #1 Solution

Vector $\vec{B}$ has a magnitude of 3 meters ($B = 3$ m) and is oriented at an angle 60° clockwise from the $-x$ axis. Find B_x and B_y.



$\theta_B = 180° - 60°$
$\theta_B = 120°$

The angle from $+x$ axis to vector $\vec{B}$ is $180° - 60° = 120°$.

$$B\cos\theta_B = B_x$$
$$(3\text{ m})\cos 120° = B_x$$
$$\boxed{-1.5\text{ m} = B_x}$$

$$B\sin\theta_B = B_y$$
$$(3\text{ m})\sin 120° = B_y$$
$$\boxed{3\frac{\sqrt{3}}{2}\text{ m} = B_y}$$

Note that you can also use the angle that $\vec{B}$ makes with the $-x$ axis. This requires **you** to assign correct signs for B_x and B_y.

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$$
$$\cos\theta = \frac{B_x}{B}$$
$$B\cos\theta = B_x$$
$$(3\text{ m})\cos 60° = B_x$$
$$1.5\text{ m} = B_x$$

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$$
$$\sin\theta = \frac{B_y}{B}$$
$$B\sin\theta = B_y$$
$$(3\text{ m})\sin 60° = B_y$$
$$3\frac{\sqrt{3}}{2}\text{ m} = B_y$$

B_x points in the **negative x direction**, so it needs a negative sign.

$$\boxed{-1.5\text{ m} = B_x} \quad \boxed{3\frac{\sqrt{3}}{2}\text{ m} = B_y}$$

$$\vec{B} = \left(-1.5\hat{\imath} + 3\frac{\sqrt{3}}{2}\hat{\jmath}\right)\text{ m}$$

Practice #2 Solution

Find the magnitude and direction of vector $\vec{B} = \left(-\frac{3}{2}\hat{\imath} + 3\frac{\sqrt{3}}{2}\hat{\jmath}\right)$ m.

$\vec{B} = B_x\hat{\imath} + B_y\hat{\jmath} + B_z\hat{k}$

$(B_x = -\frac{3}{2}\text{ m}, B_y = 3\frac{\sqrt{3}}{2}\text{ m}, B_z = 0)$

Magnitude:

Use Pythagorean Theorem

$$B = \sqrt{B_x^2 + B_y^2 + B_z^2}$$

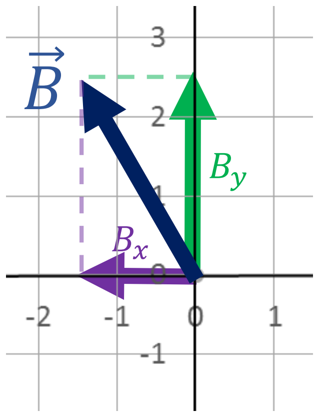
$$B = \sqrt{\left(-\frac{3}{2}\text{m}\right)^2 + \left(3\frac{\sqrt{3}}{2}\text{m}\right)^2 + (0)^2}$$

$$B = \sqrt{\frac{9}{4}\text{m}^2 + \frac{27}{4}\text{m}^2}$$

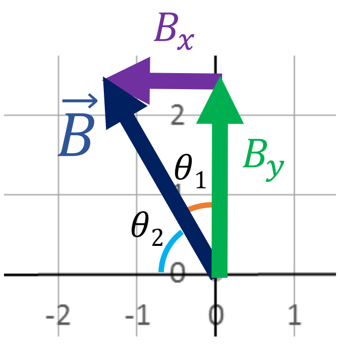
$$\boxed{B = 3\text{ m}}$$

Direction:

Step 1: Sketch the vector



Step 2: Use trigonometry to find either θ_1 or θ_2



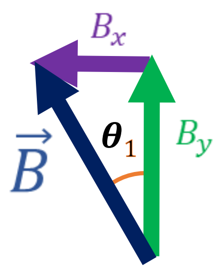
Since we ultimately want the angle from the positive x-axis, let's find θ_1

Solution continues on next page

Practice #2 Solution Continued

Find the magnitude and direction of vector $\vec{B} = \left(-\frac{3}{2}\,\hat{\imath} + 3\frac{\sqrt{3}}{2}\hat{\jmath}\right)$ m.

Step 3: Use Trigonometry to find θ_1



$$\tan\theta_1 = \frac{\text{opp}}{\text{adj}}$$

$$\tan\theta_1 = \frac{|B_x|}{|B_y|}$$

$$\tan\theta_1 = \frac{\left|-{}^{3}/_{2}\right|\text{ m}}{\left|3\frac{\sqrt{3}}{2}\right|\text{ m}}$$

$$\tan\theta_1 = \frac{({}^{3}/_{2})}{({}^{3}/_{2})\sqrt{3}}$$

$$\tan\theta_1 = \frac{1}{\sqrt{3}}$$

$$\theta_1 = \tan^{-1}\frac{1}{\sqrt{3}}$$

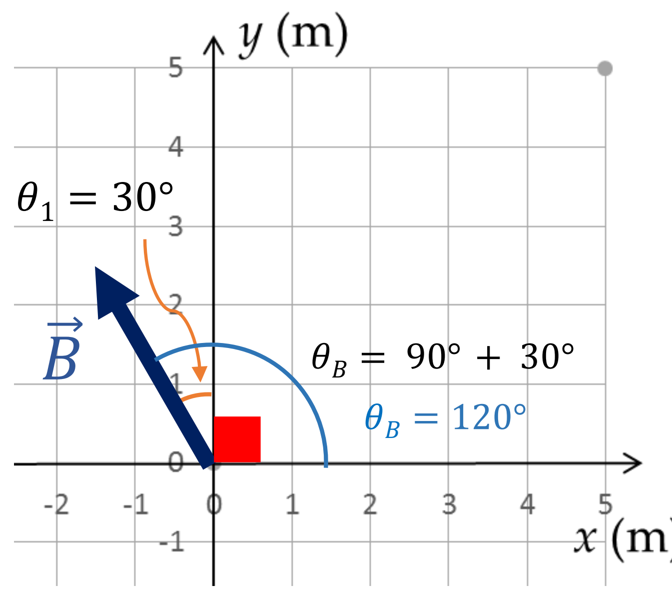
$$\theta_1 = 30^{\circ}$$

Step 4: Now that we have θ_1, we want to find the total angle from the +x axis. We add 90° to θ_1 (because the angle between the x and y axis is always 90°)

$$\theta_B = 90^{\circ} + \theta_1$$

$$\theta_B = 90^{\circ} + 30^{\circ}$$

$$\boxed{\theta_B = 120^{\circ}}$$



$$\boxed{\vec{B} = 3\text{ m}\angle 120^{\circ}}$$

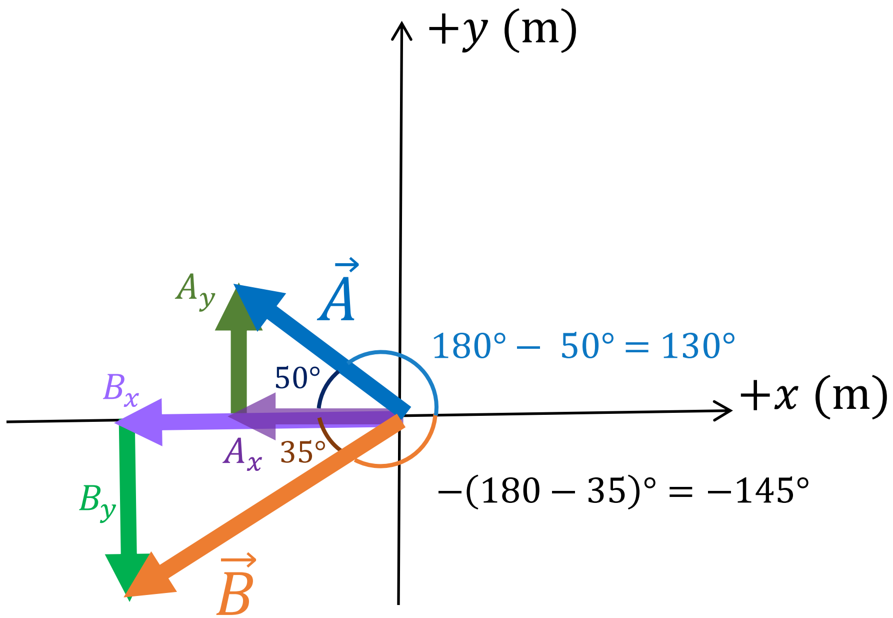
Practice #3 Solution

A = 2.5 m
50° CW from – x axis

B = 4 m
35° CCW from – x axis

Find the magnitude and direction of the vector $\vec{C} = \vec{A} + \vec{B}$.

- First, find the components of $\vec{A}$ and $\vec{B}$



$A \cos \theta_A = A_x$

$(2.5 \text{ m}) \cos(130°) = A_x$

$-1.61 \text{ m} \cong A_x$

$A \sin \theta_A = A_y$

$(2.5 \text{ m}) \sin(130°) = A_y$

$1.92 \text{ m} \cong A_y$

$B \cos \theta_B = B_x$

$(4 \text{ m}) \cos(-145°) = B_x$

$-3.28 \text{ m} \cong B_x$

$B \sin \theta_B = B_y$

$(4 \text{ m}) \sin(-145°) = B_y$

$-2.29 \text{ m} \cong B_y$

$$\vec{A} = (-1.61\hat{\imath} + 1.92\hat{\jmath} + 0\hat{k}) \text{ m}$$

$$\vec{B} = (-3.28\hat{\imath} - 2.29\hat{\jmath} + 0\hat{k}) \text{ m}$$

Solution continues on next page

Practice #3 Solution Continued

Find the magnitude and direction of the vector $\vec{C} = \vec{A} + \vec{B}$.

- First, find the components of $\vec{A}$ and $\vec{B}$

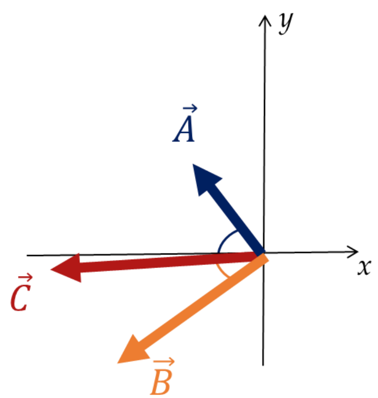
$$\vec{A} = (-1.61\hat{\imath} + 1.92\hat{\jmath} + 0\hat{k})\ \text{m}$$
$$\vec{B} = (-3.28\hat{\imath} - 2.29\hat{\jmath} + 0\hat{k})\ \text{m}$$

- Next, do the addition

$$\begin{array}{rl} \vec{A} & (-1.61\hat{\imath} + 1.92\hat{\jmath} + 0\hat{k})\ \text{m} \\ +\vec{B} & +(-3.28\hat{\imath} - 2.29\hat{\jmath} + 0\hat{k})\ \text{m} \\ \hline \vec{C} & (-4.89\hat{\imath} - 0.37\hat{\jmath} + 0\hat{k})\ \text{m} \end{array}$$

$$C_x = A_x + B_x = -4.89\ \text{m}$$
$$C_y = A_y + B_y = -0.37\ \text{m}$$

$$\vec{C} = (-4.89\hat{\imath} - 0.37\hat{\jmath} + 0\hat{k})\ \text{m}$$



- Then calculate the magnitude and direction of the resulting vector $\vec{C}$

Magnitude:

$$C = \sqrt{C_x^2 + C_y^2 + C_z^2}$$
$$= \sqrt{(-4.89)^2 + (-0.37)^2 + (0)^2}$$

$$\boxed{C = 4.9\ \text{m}}$$

Direction:

$$\tan\theta = \frac{\text{opp}}{\text{adj}}$$
$$\tan\theta = \frac{|C_y|}{|C_x|}$$

$$\tan\theta = \frac{|-0.37|}{|-4.89|}$$
$$\tan\theta = 0.0757$$
$$\theta = \tan^{-1}(0.0757)$$
$$\theta = 4.3°$$

This angle is measured CCW from the $-x$ axis.
The total angle from the $+x$ axis is $\boxed{184°}$

$$\boxed{\vec{C} = 4.9\ \text{m} \angle 184°}$$

Practice #4 Solution

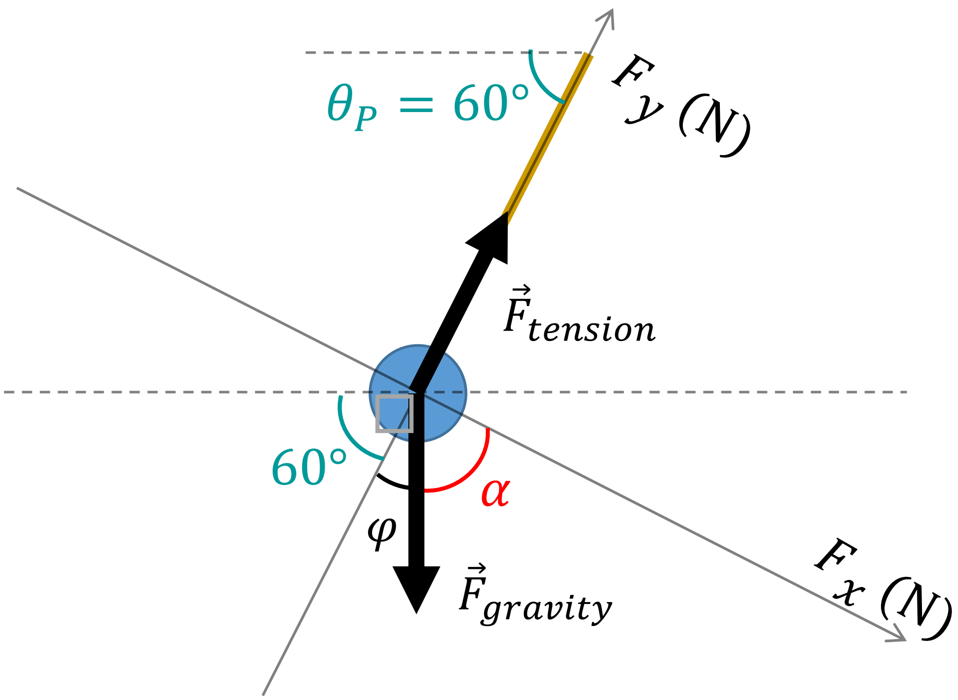
If $F_g = 6$ N, find F_{g_x} and F_{g_y}.

- Notice that 60° is the angle between the horizontal and the tilted y-axis. Since F_g points vertically downward, we know $60° + \varphi = 90°$, so $\varphi = 30°$
- Similarly, $30^o + \alpha = 90^o$

  $\alpha = 60^o$

- This angle is in the clockwise direction from the +x axis, so to get the standard direction, we give it a negative sign:

  $\alpha = -60^o$



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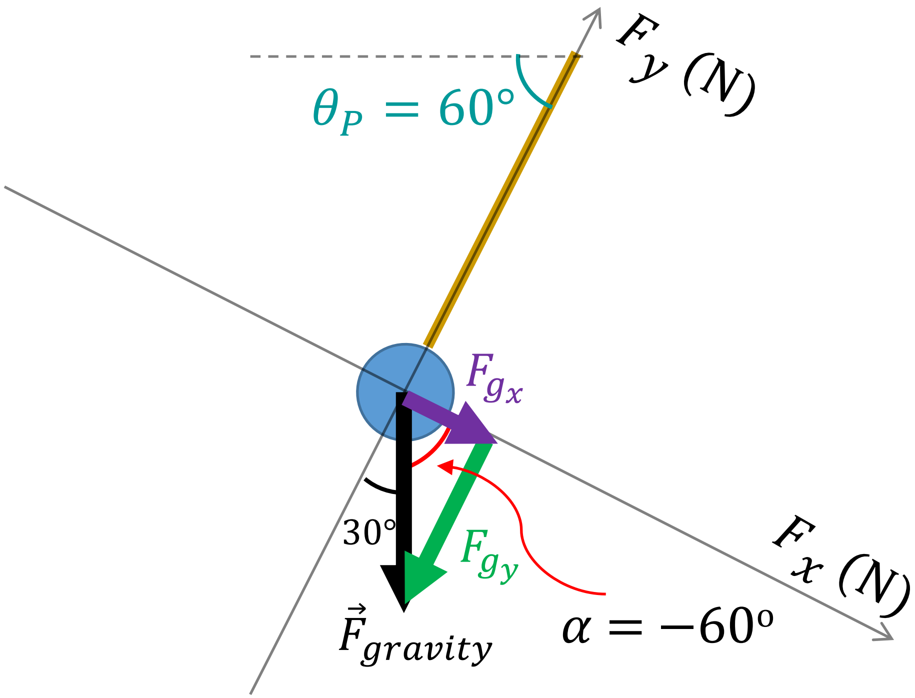
Practice #4 Solution Continued

Make sure to draw F_{g_x} and F_{g_y} **parallel** to their respective axes

$$F_g \cos \alpha = F_{g_x}$$
$$(6\text{ N})\cos(-60°) = F_{g_x}$$
$$3\text{ N} = F_{g_x}$$

$$F_g \sin \alpha = F_{g_y}$$
$$(6\text{ N})\sin(-60°) = F_{g_y}$$
$$-3\sqrt{3}\text{ N} = F_{g_y}$$



$$3\text{ N} = F_{g_x}$$
$$-3\sqrt{3}\text{ N} = F_{g_y}$$

Reminder: If we use α, measured from the $+x$ axis, the calculator gives the appropriate positive and negative signs